A MINIMAL TYPE OF THE 2-ADIC WEIL REPRESENTATION

AARON WOOD

Abstract. A minimal type of the even 2-adic Weil representation is described. The Hecke algebra of this type is isomorphic to the classical affine Hecke algebra of type B_n. In this way the Weil representation of a metaplectic group corresponds to the trivial representation of an orthogonal group in the local Shimura correspondence.

Introduction

Let $\mathcal{W}$ be a nondegenerate symplectic space of a p-adic field, let $G = \widetilde{\mathrm{Sp}}(\mathcal{W})$ be a nontrivial two-fold central extension of the symplectic group $\underline{G} = \mathrm{Sp}(\mathcal{W})$ of type C_n, and let ω be the Weil representation of G. If p is odd, ω has a one-dimensional subspace upon which the inverse image of the Iwahori subgroup acts. This minimal type was used by Wee-Tek Gan and Gordan Savin in [2] to establish an equivalence of categories between certain representations of G and certain other representations of orthogonal groups of type B_n. In this equivalence, the even Weil representation ω^e corresponds to the trivial representation of a split adjoint group of type B_n. In particular, the correspondence is realized as an isomorphism between the Hecke algebras of these types and the affine Hecke algebras of type B_n.

If $p = 2$, there are no Iwahori-fixed vectors of the Weil representation. However, for the two-fold cover $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ of $\mathrm{SL}_2(\mathbb{Q}_2)$, there is a one-dimensional type of the Weil representation for the first congruence subgroup of the Iwahori subgroup. The corresponding Hecke algebra was shown to be isomorphic to that of $\mathrm{PGL}_2(\mathbb{Q}_2)$ in [7]. The main result of this paper is to extend the result for $p = 2$ to larger symplectic groups; that is, to find a minimal type whose corresponding Hecke algebra is the affine Hecke algebra of type B_n.

The description of the appropriate open compact subgroup stems from the following fact: in characteristic 2, a symmetric quadratic form is alternating. Hence, the finite orthogonal group $\mathrm{O}_{2n}(2)$ is a subgroup of the finite symplectic group $\mathrm{Sp}_{2n}(2)$. If B' is a Borel subgroup of $\mathrm{O}_{2n}(2)$ which sits in a Borel subgroup B of $\mathrm{Sp}_{2n}(2)$, then the inverse image $\underline{K}$ of B' (under the projection map $\mathbb{Z}_2 \to \mathbb{F}_2$) is a subgroup of index 2^n of the Iwahori subgroup, as pictured in the diagram below.

Partially supported by NSF grant DMS-0852429.

$$\begin{tikzcd}
\underline{K} \arrow[r] \arrow[d] & \underline{I} \arrow[r] \arrow[d] & \mathrm{Sp}_{2n}(\mathbb{Z}_2) \arrow[d] \\
B' \arrow[r] & B \arrow[r] & \mathrm{Sp}_{2n}(2)
\end{tikzcd}$$

In the SL_2 case, this subgroup $\underline{K}$ is exactly the subgroup from [7]. Let K be the inverse image in G of $\underline{K}$.

Section 1 of this paper is dedicated to fixing notation and summarizing previously known, but relevant, results. In Section 2, the $\widetilde{\mathrm{SL}}_2$ case is worked out in detail. Specifically, the one-dimensional subspace on which K acts is the span of the characteristic function of $\mathbb{Z}_2$. Since K is an index 2 subgroup of the Iwahori subgroup, the support of the associated Hecke algebra $\mathcal{H}$ is potentially much larger than that of the classical Iwahori-Hecke algebra. However, it is shown that $\mathcal{H}$ is supported exactly on those K-double cosets parametrized by the affine Weyl group. Using some general facts about Hecke algebras and finite-index subgroups, generators T_0 and T_1 for $\mathcal{H}$ are given which satisfy only the quadratic relations

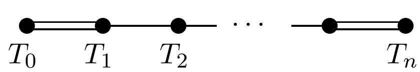
$$(T_0 - 2)(T_0 + 1) = 0, \qquad \text{and} \qquad (T_1 - 1)(T_1 + 1) = 0.$$

These are exactly the quadratic relations for the affine Hecke algebra of $\mathrm{PGL}_2(\mathbb{Q}_2)$.

In Section 3, the general symplectic case is described. In this setting, the one-dimensional K-type is the span of the characteristic function of the standard lattice, and the support of the associated Hecke algebra $\mathcal{H}$ is again reduced to those K-double cosets parametrized by the affine Weyl group. Similar to, and drawing from, the $\widetilde{\mathrm{SL}}_2$ case, generators $T_0, \ldots, T_n$ for $\mathcal{H}$ are defined which satisfy the quadratic relations

$$(T_i - 2)(T_i + 1) = 0, \qquad \text{and} \qquad (T_n - 1)(T_n + 1) = 0.$$

The braid relations for these generators are given by the following Coxeter diagram.



The affine Hecke algebra A of type B_n has generators $t_0, \ldots, t_n$ which satisfy the quadratic relations $(t_i - 2)(t_i + 1) = 0$. The Coxeter diagram of type B_n has an involution given by $\tau t_1 \tau = t_0$, which satisfies $\tau^2 = 1$ and $\tau t_1 \tau t_1 = t_1 \tau t_1 \tau$. The isomorphism between $\mathcal{H}$ and A is given by

$$t_n \mapsto T_0, \quad \ldots \quad t_1 \mapsto T_{n-1} \quad \text{and} \quad \tau \mapsto T_n.$$

This paper is a part of the author's Ph.D. thesis. The author would like to thank his advisor Gordan Savin for his invaluable guidance throughout this endeavor.

1. Preliminaries

In this section, notation will be fixed and some previously known, but relevant, results will be summarized.

1.1. **Symplectic Vector Spaces.**

Let $\mathcal{W}$ be a $2n$-dimensional vector space over a field F, and let Q be a nondegenerate, skew-symmetric form on $\mathcal{W}$. Such a form Q is called a *symplectic form* and such a vector space $\mathcal{W}$ is called a *symplectic space*.

A subspace $\mathcal{X}$ of $\mathcal{W}$ is *isotropic* if Q is identically zero on $\mathcal{X}$. For a maximal isotropic subspace $\mathcal{X}$ of $\mathcal{W}$, there is a complementary subspace $\mathcal{Y}$ which is also a maximal isotropic subspace of $\mathcal{W}$, and each of $\mathcal{X}, \mathcal{Y}$ is a vector subspace of dimension n. Such a decomposition $\mathcal{W} = \mathcal{X} + \mathcal{Y}$ into maximal isotropic subspaces is called a *complete polarization* of $\mathcal{W}$.

Let $\mathcal{X} + \mathcal{Y}$ be a complete polarization of $\mathcal{W}$ and let $\{e_i\}$ be a basis of $\mathcal{X}$. There is a basis $\{f_i\}$ of $\mathcal{Y}$ such that $Q(e_i, f_j) = \delta_{ij}$. The resulting basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of $\mathcal{W}$ is called a *symplectic basis*. Under such a basis, elements of $\mathcal{W}$ may be considered as column vectors and the symplectic form Q may be expressed as

$$Q(u, v) = {}^{\mathsf{T}}u J v,$$

where ${}^{\mathsf{T}}u$ denotes the transpose of u and J is the $2n \times 2n$ matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

1.2. **Symplectic Lie Algebras.**

Let $\mathcal{W}$ be a $2n$-dimensional symplectic space over $\mathbb{C}$ with symplectic form Q. The *symplectic Lie algebra*, $\mathfrak{sp}(\mathcal{W})$, is defined to be the Lie algebra of linear endomorphisms T of $\mathcal{W}$ satisfying

$$Q(Tu, v) + Q(u, Tv) = 0$$

for all u, v in $\mathcal{W}$. The set $\mathfrak{sp}(\mathcal{W})$ is a vector space under the usual addition and scalar multiplication of linear operators and an algebra under the Lie bracket

$$[T_1, T_2] = T_1 T_2 - T_2 T_1.$$

Under the symplectic basis, $\mathfrak{sp}(\mathcal{W})$ becomes the Lie subalgebra of $\mathfrak{gl}_{2n}(\mathbb{C})$ given by

$$\mathfrak{sp}(\mathcal{W}) = \left\{ \begin{bmatrix} a & b \\ c & -{}^{\mathsf{T}}a \end{bmatrix} \in \mathfrak{gl}_{2n}(\mathbb{C}) : b = {}^{\mathsf{T}}b,\ c = {}^{\mathsf{T}}c \right\},$$

where the ${}^{\mathsf{T}}a$ denotes the transpose of the matrix a.

1.2.1. *Cartan Decomposition.*

Let $\mathfrak{h}$ be the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{sp}(\mathcal{W})$ and let $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ be its linear dual. The Cartan subalgebra is n-dimensional and an arbitrary element H of $\mathfrak{h}$ is of the form

$$H = \begin{bmatrix} a & \\ & -a \end{bmatrix},$$

where a is a diagonal matrix with entries $a_1, \ldots, a_n$. The dual basis of $\mathfrak{h}^*$ is $\lambda_1, \ldots, \lambda_n$, defined by $\lambda_i(H) = a_i$.

Let E_{ij} be the $n \times n$ matrix with a 1 in the ijth position and 0 elsewhere. The one-dimensional subspaces of $\mathfrak{sp}(\mathcal{W})$ that lie outside of $\mathfrak{h}$ are spanned by the following matrices (where $i \neq j$):

$$\begin{bmatrix} E_{ji} & 0 \\ 0 & -E_{ij} \end{bmatrix}, \quad \begin{bmatrix} 0 & E_{ii} \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ E_{ii} & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{bmatrix}.$$

For the $H \in \mathfrak{h}$, the Lie bracket $[H, X]$ is given by

$$\begin{array}{rcrl} (a_i - a_j)X & = & (\lambda_i - \lambda_j)(H)X & \text{if } X \text{ is of the first type,} \\ 2a_iX & = & 2\lambda_i(H)X & \text{if } X \text{ is of the second type,} \\ -2a_iX & = & -2\lambda_i(H)X & \text{if } X \text{ is of the third type,} \\ (a_i + a_j)X & = & (\lambda_i + \lambda_j)(H)X & \text{if } X \text{ is of the fourth type,} \\ -(a_i + a_j)X & = & -(\lambda_i + \lambda_j)(H)X & \text{if } X \text{ is of the fifth type.} \end{array}$$

Therefore, the roots in $\mathfrak{h}^*$ are

$$\Phi = \big\{ \pm \lambda_i \pm \lambda_j : i \neq j \big\} \cup \big\{ \pm 2\lambda_i \big\}.$$

The roots in the first set are called *short roots* and those in the second set are called *long roots*. If the elementary matrix corresponding to a root α is denoted by X_α, then the root space of α is $\mathbb{C}X_\alpha$, and the Cartan decomposition is

$$\mathfrak{sp}(\mathcal{W}) = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathbb{C}X_\alpha.$$

1.2.2. *Roots.* Consider the real vector space $\mathfrak{h}^*_{\mathbb{R}}$, defined to be the $\mathbb{R}$-span of $\{\lambda_i\}$. This vector space is a Euclidean space under the usual dot product, denoted $(\ ,\ )$, which is symmetric and positive definite. This inner product is the same as the one which arises from the Killing form on $\mathfrak{h}$ (see [9]). For convenience, if $\mu, \lambda \in \mathfrak{h}^*_{\mathbb{R}}$, write

$$\langle \mu, \lambda \rangle = \frac{2(\mu, \lambda)}{(\lambda, \lambda)}.$$

For each $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$, define the reflection s_λ on $\mathfrak{h}^*_{\mathbb{R}}$ by

$$s_\lambda(\mu) = \mu - \langle \mu, \lambda \rangle \lambda.$$

The proof of the following proposition can be found in [1].

Proposition 1.1. *The set of roots Φ forms a root system in $\mathfrak{h}^*_{\mathbb{R}}$, and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ forms a set of simple roots in Φ, where $\alpha_i = \lambda_i - \lambda_{i+1}$, for $i = 1, \ldots, n-1$, and $\alpha_n = 2\lambda_n$. The corresponding set of positive roots is $\Phi^+ = \big\{\lambda_i \pm \lambda_j : i < j\big\} \cup \big\{2\lambda_i\big\}$.*

For $i = 1, \ldots, n$, define s_i to be the reflection s_{α_i}. The Weyl group W of $\mathfrak{sp}(\mathcal{W})$ is the finite group of reflections s_α for α in Φ. It is a Coxeter group and is generated by the set of simple reflections $\{s_1, \ldots, s_n\}$. The braid relations for the generators of W are given by the following Coxeter diagram (see [3]).

1.2.3. *Affine Roots.* The set of affine roots is the set

$$\Phi^{\mathrm{aff}} = \{\alpha + n : \alpha \in \Phi, n \in \mathbb{Z}\}$$

of affine functionals on $\mathfrak{h}$ which act by $(\alpha + n)(H) = \alpha(H) + n$. Let α_* be the highest root in Φ given by

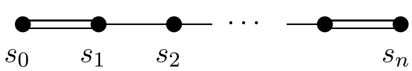
$$\alpha_* = 2\lambda_1 = 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n,$$

and define the affine root α_0 by

$$\alpha_0 = 1 - \alpha_*.$$

For each affine root γ, the affine reflection s_γ is the reflection in $\mathfrak{h}$ across the hyperplane $\{H : s_\gamma(H) = 0\}$. Define s_0 to be the affine reflection $s_0 = s_{\alpha_0}$.

The group W^{aff} of affine reflections is called the affine Weyl group; it is a Coxeter group and is generated by the simple affine reflections $\{s_0, s_1, \ldots, s_n\}$. The braid relations for these generators of W^{aff} are given by the following Coxeter diagram (see [3]).



1.2.4. *Chevalley Basis.* For each $\alpha \in \Phi^+$ define $H_\alpha = [X_\alpha, X_{-\alpha}]$. More explicitly,

$$\text{if } X_\alpha = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix}, \quad \text{then} \quad H_\alpha = \begin{bmatrix} E_{ii} - E_{jj} & 0 \\ 0 & -E_{ii} + E_{jj} \end{bmatrix};$$

$$\text{if } X_\alpha = \begin{bmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{bmatrix}, \quad \text{then} \quad H_\alpha = \begin{bmatrix} E_{ii} + E_{jj} & 0 \\ 0 & -E_{ii} - E_{jj} \end{bmatrix};$$

$$\text{if } X_\alpha = \begin{bmatrix} 0 & E_{ii} \\ 0 & 0 \end{bmatrix}, \quad \text{then} \quad H_\alpha = \begin{bmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{bmatrix}.$$

The proof of the following proposition can be found in [1] or [9].

Proposition 1.2. *The following properties hold for all $\alpha, \beta \in \Phi$.*

1. $[H_\alpha, H_\beta] = 0$.
2. $[H_\alpha, X_\beta] = \langle \beta, \alpha \rangle X_\beta$.
3. $[X_\alpha, X_\beta] = \begin{cases} H_\alpha & \text{if } \beta = -\alpha, \\ c(\alpha, \beta) X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$

   where $c(\alpha, \beta)$ is either ± 1 or ± 2.

In particular, $\{X_\alpha : \alpha \in \Phi\} \cup \{H_\alpha : \alpha \in \Pi\}$ *is a Chevalley basis for* $\mathfrak{sp}(\mathcal{W})$.

1.2.5. *Universal Enveloping Algebra.* Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{sp}(\mathcal{W})$. The Poincaré-Birkhoff-Witt Theorem implies that $\mathfrak{sp}(\mathcal{W})$ can be embedded into $\mathcal{U}$ and that, if $\{Y_1, \ldots, Y_r\}$ is an ordering of the basis $\{X_\alpha : \alpha \in \Phi\} \cup \{H_\alpha : \alpha \in \Pi\}$ of $\mathfrak{sp}(\mathcal{W})$, then the set of monomials $Y_1^{m_1} \cdots Y_r^{m_r}$ form a basis for $\mathcal{U}$ (see [4]).

1.3. **Symplectic Groups.**

1.3.1. *Classical Symplectic Groups.* Let $\mathcal{W}$ be a symplectic space of dimension $2n$ over a field F with symplectic form Q. The *symplectic group* $\mathrm{Sp}(\mathcal{W})$, is defined to be the group of linear automorphisms of $\mathcal{W}$ that preserve the symplectic form, i.e., those linear operators $T : \mathcal{W} \to \mathcal{W}$ such that for all u, v in $\mathcal{W}$,

$$Q(Tu, Tv) = Q(u, v).$$

Under a symplectic basis, $\mathrm{Sp}(\mathcal{W})$ becomes a matrix group. If x is the matrix corresponding to a transformation T, with

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$^{\intercal}ad - {}^{\intercal}cb = 1, \qquad {}^{\intercal}ac = {}^{\intercal}ca, \qquad \text{and} \qquad {}^{\intercal}bd = {}^{\intercal}db.$$

If $n = 1$, then the only condition is $ad - bc = 1$, in which case the symplectic group is $\mathrm{SL}_2(F)$.

1.3.2. *Symplectic Chevalley Groups.* Following [9], the classical symplectic group can be constructed from the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$. Let $\mathcal{W}$ be a symplectic vector space over $\mathbb{C}$, let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{sp}(\mathcal{W})$, and let $\mathcal{U}_\mathbb{Z}$ be the subalgebra of $\mathcal{U}$ generated by the divided powers $X_\alpha^m/m!$. Under the natural representation of $\mathfrak{sp}(\mathcal{W})$, and hence of $\mathcal{U}$, on $\mathcal{W}$ elements of $\mathcal{U}$ may be viewed as elements of the matrix algebra. In this setting, each $X_\alpha^2 = 0$, so $\mathcal{U}_\mathbb{Z}$ is generated by 1 and $\{X_\alpha : \alpha \in \Phi\}$. Therefore, since $\mathcal{W}$ is the natural representation, the standard lattice

$$L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \oplus \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_n$$

is the $\mathcal{U}_\mathbb{Z}$-invariant lattice guaranteed by Corollary 1 of Theorem 2 in [9].

Fix a field F; for an element t in F and a root α in Φ, one obtains the natural action of

$$x_\alpha(t) = \exp(tX_\alpha) = \sum_{n=0}^{\infty} t^n \frac{X_\alpha^n}{n!} = 1 + tX_\alpha$$

on $\mathcal{W}_F = L \otimes_\mathbb{Z} F$, which is the F-span of $e_1, \ldots, e_n, f_1, \ldots, f_n$. In other words, $x_\alpha(t)$ may be interpreted as the actual matrix $1 + tX_\alpha$ with entries in F.

For the data $\big(\mathfrak{sp}(\mathcal{W}), \mathcal{W}, F\big)$, one obtains a Chevalley group G, generated by

$$\{x_\alpha(t) : t \in F, \alpha \in \Phi\}.$$

This group G is exactly the classical symplectic group described above (see [9]).

The following elements play an important role in the theory of Chevalley groups. For $t \in F^\times$, define

$$\begin{aligned} w_\alpha(t) &= x_\alpha(t)x_{-\alpha}(-1/t)x_\alpha(t),\\ h_\alpha(t) &= w_\alpha(t)w_\alpha(-1). \end{aligned}$$

In addition, define the elements $w_1, \ldots, w_n$ of G by

$$w_i = w_{\alpha_i}(1),$$

where $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is the set of simple roots. The elements $w_1, \ldots, w_n$ form a set of representatives of generators of the Weyl group W in G (see [9] or [1]).

Suppose now that F is a nonarchimedian local field with uniformizer ϖ and ring of integers $\mathcal{O}$. For any affine root $\gamma = \alpha + n$, where α is a root and n is an integer, define the element

$$x_\gamma(t) = x_\alpha(\varpi^n t)$$

define G_γ to be the subgroup of G generated by $\{x_\gamma(t), x_{-\gamma}(t) : t \in F\}$, and define K_γ to be the subgroup generated by $\{x_\gamma(t), x_{-\gamma}(t) : t \in \mathcal{O}\}$. Consider the map $e_\gamma : \mathrm{SL}_2(F) \to G$ given by

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mapsto x_\gamma(t), \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \mapsto x_{-\gamma}(t).$$

Under e_γ, the image of $\mathrm{SL}_2(F)$ is G_γ and the image of $\mathrm{SL}_2(\mathcal{O})$ is K_γ.

In addition, for the affine root $\gamma = \alpha + n$, define the element

$$w_\gamma(t) = w_\alpha(\varpi^n t)$$

of G_γ. For the simple affine root $\alpha_0 = 1 - \alpha_*$ from Section 1.2.3, define the element w_0 of G to be

$$w_0 = w_{\alpha_0}(1) = w_{\alpha_*}(\varpi^{-1}).$$

The set $\{w_0, w_1, \ldots, w_n\}$ is a set of representatives in G of the simple reflections $\{s_0, s_1, \ldots, s_n\}$ that generate the affine Weyl group (see [5]).

1.3.3. *Generators and Relations.* A summary of the results on generators and relations in [9] for the symplectic Chevalley group G is the following. First, G, which is generated by the $x_\alpha(t)$, satisfies the following relations: for any $\alpha, \beta \in \Phi$ and $t, u \in F$ (nonzero where necessary),

(R1) $x_\alpha(t)x_\alpha(u) = x_\alpha(t+u)$;
(R2) $\big(x_\alpha(t), x_\beta(u)\big) = \prod x_{i\alpha+j\beta}(c_{ij}t^iu^j)$, for $\alpha + \beta \neq 0$;
(R3) $w_\alpha(t)x_\alpha(u)w_\alpha(-t) = x_{-\alpha}(-t^{-2}u)$;
(R4) $w_\alpha(1)x_\beta(t)w_\alpha(-1) = x_{s_\alpha(\beta)}(ct)$;

(R5) $w_\alpha(1)h_\beta(t)w_\alpha(-1) = h_{s_\alpha(\beta)}(t)$;
(R6) $h_\alpha(t)x_\beta(u)h_\alpha(t)^{-1} = x_\beta\big(t^{\langle\beta,\alpha\rangle}u\big)$;
(R7) $h_\alpha(t)h_\alpha(u) = h_\alpha(tu)$.

In (R2), the product is taken over the roots that are positive linear combinations of α and β, and $c_{ij} = c_{ij}(\alpha,\beta)$ is ±1 if the angle between the roots is $3\pi/4$ and ±2 if the angle is $\pi/2$. In (R4), $c = c(\alpha,\beta) = c(\alpha,-\beta) = \pm1$.

Second, relations (R1) through (R6) are consequences of (R1) and (R2) if $G = \mathrm{Sp}_{2n}(F)$ for $n \geq 2$ or of (R1) and (R3) if $G = \mathrm{SL}_2(F)$. Hence, these two relations along with relation (R7) form a complete set of relations.

1.3.4. *Central Extensions.* A group G' is a *central extension* of G if there is a surjective homomorphism from G' to G whose kernel lies in the center of G'. A central extension E of G is *universal* if it a central extension of any other central extension of G. By Theorem 10 of [9], a universal central extension E of the symplectic Chevalley group G exists and is the group defined abstractly using only relations (R1) and (R2). (In the $\mathrm{SL}_2(F)$ case, the universal central extension is abstractly defined using only relations (R1) and (R3).) Hence, the relations (R1) through (R6) can be lifted to any central extension of G.

The preceding paragraph implies that the elements $x_\alpha(t)$ lift *uniquely* to elements $x'_\alpha(t)$ in any central extension G'. Therefore, $w_\alpha(t)$ and $h_\alpha(t)$ lift *canonically* to $w'_\alpha(t)$ and $h'_\alpha(t)$ in G' via the formulas

$$\begin{aligned} w'_\alpha(t) &= x'_\alpha(t)x'_{-\alpha}(-1/t)x'_\alpha(t), \\ h'_\alpha(t) &= w'_\alpha(t)w'_\alpha(-1). \end{aligned}$$

The relation (R7) is the only relation of G which does not hold in G', so the symbol (t,u) given by $h'_\alpha(t)h'_\alpha(u) = (t,u)h'_\alpha(tu)$ provides the necessary information about this central extension.

1.4. **Induced Representations and Hecke Algebras.** Let H be a finite index subgroup of an arbitrary group G, and suppose that G admits a left Haar measure normalized to give $\mathrm{vol}(H) = 1$. The space of complex-valued functions on G will be denoted $C(G)$. Fix a complex representation (π, V) of G and a character $\chi : H \to \mathbb{C}^\times$ of H. The objects of interest are

1. the induced representation $(\sigma, U) = \mathrm{ind}_H^G \chi$, where σ acts by right translation on the vector space
$$U = \{\phi \in C(G) : \phi(hg) = \chi(h)\phi(g) \text{ for } h \in H,\, g \in G\};$$
2. the subspace $V^{H,\bar\chi}$ of V given by
$$V^{H,\bar\chi} = \big\{v \in V : \pi(h)v = \bar\chi(h)v \text{ for } h \in H\big\};$$
3. the Hecke Algebra $\mathcal{H} = \mathcal{H}(G/\!\!/H; \chi)$ defined by
$$\mathcal{H} = \big\{f \in C(G) : f(h_1gh_2) = \chi(h_1)f(g)\chi(h_2) \text{ for } h_1, h_2 \in H,\, g \in G\big\}.$$

The proofs of the following two propositions are straight-forward.

Proposition 1.3. *The spaces* $\mathrm{Hom}_H(\pi, \chi)$ *and* $\mathrm{Hom}_G(\pi, \sigma)$ *are isomorphic.*

Proposition 1.4. *The space* $V^{H,\bar{\chi}}$ *is an* $\mathcal{H}$*-module under the action*

$$\pi(f)v = \int_G f(g)\big(\pi(g)v\big)\,dg.$$

Proposition 1.5. *The Hecke algebra* $\mathcal{H}$ *is isomorphic to the endomorphism ring of* U*. Moreover, elements of* $\mathcal{H}$ *not supported on* H *act on* U *as trace-zero endomorphisms.*

Proof. Define a map $\mathcal{H} \to \mathrm{End}(U)$ by $f \mapsto T$ with $T(\phi) = f * \phi$. $T(\phi)$ is indeed an element of U since

$$T(\phi)(hg) = \int_G f(x)\phi(x^{-1}hg)dx = \int_G f(yh)\phi(y^{-1}g)dy = \chi(h)T(\phi)(g).$$

Define a map $\mathrm{End}(U) \to \mathcal{H}$ by $T \mapsto f$ with $f(g) = T(\dot{\chi})(g)$, where $\dot{\chi}$ is the extension of χ by zero to all of G. This f is an element of $\mathcal{H}$, since $\dot{\chi}$ being an element of $U^{H,\chi}$ implies that

$$f(h_1gh_2) = T(\dot{\chi})(h_1gh_2) = \chi(h_1)\sigma(h_2)\big(T(\dot{\chi})(g)\big) = \chi(h_1)f(g)\chi(h_2).$$

The composition $\mathcal{H} \to \mathrm{End}(U) \to \mathcal{H}$ of these two maps is the identity, since

$$T(\dot{\chi})(g) = \int_G f(x)\dot{\chi}(x^{-1}g)dx = \int_H f(x^{-1}g\chi(x)dg = f(g).$$

Suppose that $f \in \mathcal{H}$ is not supported on H. Since the functions $\mathrm{char}(Hx)$, for $x \in H\backslash G$, form a basis of U, to check that the trace of f is zero it suffices to see that $(f * \phi)(x) = 0$ for $\phi \in U$ with $\mathrm{supp}(\phi) = Hx \neq H$. Indeed, for such a ϕ,

$$(f * \phi)(x) = \int_G f(g)\phi(g^{-1}x)dg = \begin{cases} \displaystyle\int_G 0 \cdot \phi(g^{-1}x)dg = 0 & \text{if } g \in H, \\ \displaystyle\int_G f(g) \cdot 0\; dg = 0 & \text{if } g \notin H. \end{cases}$$

□

These two $\mathcal{H}$-modules are compatible in the following way. Let $V_1, \dots, V_n$ be the irreducible representations of G such that $V_i^{H,\bar{\chi}}$ is nontrivial. The matrix coefficients $v_i^*(\pi_i(g^{-1})v_i)$, for $v_i^* \in V_i^*$ and $v_i \in V_i$, give a part $\bigoplus V_i \otimes V_i^*$ of the decomposition of $C(G)$, and hence,

$$U^{H,\bar{\chi}} = \bigoplus_{i=1}^{n} V_i^{H,\bar{\chi}} \otimes V_i^*.$$

In particular, if $\phi(g) = v_i^*(\pi_i(g^{-1})v_i)$, then

$$\begin{aligned}(f * \phi)(x) &= \int_G f(g)v_i^*\big(\pi_i(x^{-1}g)v_i\big)dg \\ &= v_i^*\big(\pi_i(x^{-1})(\pi_i(f)v_i)\big) \\ &= \pi_i(f)(v_i \otimes v_i^*).\end{aligned}$$

1.5. **Characters of $\mathbb{Q}_p$.** Let ψ be a nontrivial, smooth, additive character of $\mathbb{Q}_p$. As ψ is locally constant, there exists a smallest integer c , called the *conductor* of ψ, such that ψ is trivial on $p^c\mathbb{Z}_p$. The proof of the following proposition can be found in [10].

Proposition 1.6. *Fix a nontrivial smooth character ψ of $\mathbb{Q}_p$. Every smooth character of $\mathbb{Q}_p$ is of the form $x \mapsto \psi(ax)$ for some a in $\mathbb{Q}_p$.*

There is a natural projection from $\mathbb{Q}_p$ to $\mathbb{Q}_p/\mathbb{Z}_p$, a canonical embedding of $\mathbb{Q}_p/\mathbb{Z}_p$ into the p-torsion of $\mathbb{Q}/\mathbb{Z}$, and a nearly canonical embedding of $\mathbb{Q}/\mathbb{Z}$ into $\mathbb{C}^\times$ given by $x \mapsto e^{2\pi ix}$. The composition of these maps

$$\mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$$

defines an additive character $\psi_1 : \mathbb{Q}_p \to \mathbb{C}^\times$ of conductor $c = 0$.

For a in $\mathbb{Q}_p^\times$, define the character $\psi_a : \mathbb{Q}_p \to \mathbb{C}^\times$ by

$$\psi_a(x) = \psi_1(ax),$$

which has conductor $c = -\operatorname{val}(a)$. According to Proposition 1.6, these ψ_a account for all nontrivial, smooth, additive characters of $\mathbb{Q}_p$.

1.6. **Fourier Transform.** Let $S(\mathbb{Q}_p)$ denote the set of Schwarz functions on $\mathbb{Q}_p$, i.e., the set of smooth, compactly supported complex valued functions on $\mathbb{Q}_p$, and fix ψ to be an additive character of $\mathbb{Q}_p$ of conductor c. The Fourier transform (with respect to ψ) on $S(\mathbb{Q}_p)$ is given by $\phi \mapsto \widehat{\phi}$ where

$$\widehat{\phi}(y) = \int_{\mathbb{Q}_p} \psi(2uy)\phi(u)du.$$

For the given ψ, the Haar measure on $\mathbb{Q}_p$ will be normalized so that $\widehat{\widehat{\phi}}(y) = \phi(-y)$. The appearance of a 2 in the Fourier transform affects the role of the conductor of ψ for $p = 2$, so it will be convenient to define

$$\delta = \begin{cases} 1 & \text{if } p = 2 \\ 0 & \text{if } p \neq 2. \end{cases}$$

Proposition 1.7. *Suppose that ϕ is supported on $p^m\mathbb{Z}_p$ and constant on $p^n\mathbb{Z}_p$-cosets for $m \leq n$. Then $\widehat{\phi}$ is supported on $p^{-n+(c-\delta)}\mathbb{Z}_p$ and constant on $p^{-m+(c-\delta)}\mathbb{Z}_p$-cosets.*

Proof. Since ϕ is constant on $p^n\mathbb{Z}_p$-cosets,

$$\begin{aligned} \widehat{\phi}(y) &= \int_{\mathbb{Q}_p} \psi(2uy)\phi(u)du \\ &= \int_{\mathbb{Q}_p} \psi(2uy)\phi(u + p^n)du \\ &= \int_{\mathbb{Q}_p} \psi(2(v - p^n)y\phi(v)dv \\ &= \psi(-2p^n y)\widehat{\phi}(y) \end{aligned}$$

so $\widehat{\phi}(y) \neq 0$ if and only if $y \in p^{-n+(c-\delta)}\mathbb{Z}_p$.

The Fourier transform $\widehat{\phi}$ is constant on $p^r\mathbb{Z}_p$-cosets if and only if r is the smallest integer such that $\psi(2p^r u) = 1$ for all $u \in p^m\mathbb{Z}_p$, which happens if and only if $r = -m + (c - \delta)$. □

Proposition 1.8. *For $m \in \mathbb{Z}$, denote the characteristic function of $p^m\mathbb{Z}_p$ by ϕ_m. Then*

$$\widehat{\phi}_m = p^{-m+(c-\delta)/2}\phi_{-m+(c-\delta)}$$

Proof. The Fourier transform of $\widehat{\phi}_m$ is given by

$$\widehat{\phi}_m(y) = \int_{p^m\mathbb{Z}_p} \psi(2uy)du = \begin{cases} \mathrm{vol}(p^m\mathbb{Z}_p) & \text{if } y \in p^{-m+(c-\delta)}\mathbb{Z}_p \\ 0 & \text{otherwise.} \end{cases}$$

The normalization of the Haar measure on $\mathbb{Q}_p$ gives

$$\phi_0 = \widehat{\widehat{\phi}}_0 = \mathrm{vol}(\mathbb{Z}_p)\widehat{\phi}_{c-\delta} = \mathrm{vol}(\mathbb{Z}_p)\,\mathrm{vol}(p^{c-\delta}\mathbb{Z}_p)\phi_0 = p^{-(c-\delta)}\,\mathrm{vol}(\mathbb{Z}_p)^2\phi_0,$$

so the measure (with respect to ψ) of $\mathbb{Z}_p$ is $p^{-(c-\delta)/2}$. □

Corollary 1.9. *If ψ is a character of conductor $c = \delta$, then the Fourier transform with respect to ψ is given by $\widehat{\phi}_m = p^{-m}\phi_{-m}$.*

1.6.1. *Functions on $\mathbb{Q}_p^n$.* Suppose that $\mathcal{Y}$ is a vector space over $\mathbb{Q}_p$ and consider elements of $\mathcal{Y}$ as column vectors with respect to some basis. Let $S(\mathcal{Y})$ denote the set of Schwarz functions on $\mathcal{Y}$ and fix an additive character ψ of $\mathbb{Q}_p$. The Fourier transform with respect to ψ on $S(\mathcal{Y})$ is defined by $\phi \mapsto \widehat{\phi}$ where

$$\widehat{\phi}(y) = \int_{\mathcal{Y}} \psi(2\,{}^{\intercal}uy)\phi(u)du.$$

The Haar measure on $\mathcal{Y}$ is also normalized to give $\widehat{\widehat{\phi}}(-y) = \phi(y)$.

1.7. **Weil Representation.** Let F be a nonarchimedian local field with absolute value $|\cdot|$, let $\mathcal{W}$ be a $2n$-dimensional symplectic space over F with symplectic form Q, and fix an additive character ψ of F.

The Heisenberg group $H(\mathcal{W})$ is defined to be the set $\mathcal{W} \times F$ with group multiplication

$$(u, s) \cdot (v, t) = \big(u + v\,,\ s + t + Q(u, v)\big).$$

The center of $H(\mathcal{W})$ is $Z = \{(0, t)\} = F$; indeed, $(v, t) \in Z$ if and only if $Q(u, v) = 0$ for every u in $\mathcal{W}$. Since the symplectic group $\mathrm{Sp}(\mathcal{W})$ preserves Q, it acts on the Heisenberg group by

$$g(v, t) = (gv, t).$$

Let (ρ, S) be a representation of $H(\mathcal{W})$ with central character ψ; that is, $\rho(0, t)h = \psi(t)h$. One can twist ρ by $g \in \mathrm{Sp}(\mathcal{W})$ to obtain the representation (ρ^g, S) given by

$$\rho^g(v, t) = \rho\big(g(v, t)\big) = \rho(gv, t),$$

which also has central character ψ. By the Stone-von Neumann theorem, ρ^g must be isomorphic to ρ, so for each $g \in \mathrm{Sp}(\mathcal{W})$, there exists an operator $T(g) : S \to S$, unique up to a scalar in $\mathbb{C}^\times$, which intertwines the two representations; that is,

$$T(g)\rho = \rho^g T(g).$$

This operator $T : \mathrm{Sp}(\mathcal{W}) \to \mathrm{GL}(S)/\mathbb{C}^\times$ is a projective representation of $\mathrm{Sp}(\mathcal{W})$. Define a subgroup $\widetilde{\mathrm{Sp}}(\mathcal{W})$ of $\mathrm{Sp}(\mathcal{W}) \times \mathrm{GL}(S)$ by

$$\widetilde{\mathrm{Sp}}(\mathcal{W}) = \Big\{ \big(g, T(g)\big) : T(g)\rho = \rho^g T(g) \Big\}.$$

This group is called a metaplectic group and fits into the exact sequence

$$1 \to \mathbb{C}^\times \to \widetilde{\mathrm{Sp}}(\mathcal{W}) \to \mathrm{Sp}(\mathcal{W}) \to 1.$$

$\widetilde{\mathrm{Sp}}(\mathcal{W})$ is a central extension of $\mathrm{Sp}(\mathcal{W})$ by $\mathbb{C}^\times$, so T lifts uniquely (see [9]) to a linear representation $\omega : \widetilde{\mathrm{Sp}}(\mathcal{W}) \to \mathrm{GL}(S)$, called the *Weil representation* with respect to ψ.

1.7.1. *Models.* For any closed subgroup $\mathcal{Z}$ of $\mathcal{W}$ define

$$\mathcal{Z}^\perp = \big\{ v \in \mathcal{W} : \psi\big(Q(v,z)\big) = 1 \text{ for all } z \in \mathcal{Z} \big\},$$

which is also a closed subgroup of $\mathcal{W}$. Define $H(\mathcal{Z})$ to be the subgroup $\mathcal{Z} \times F$ of the Heisenberg group, and assume that $\mathcal{Z} \subset \mathcal{Z}^\perp$. The character ψ can be extended trivially to $H(\mathcal{Z})$ by $\psi(z,t) = \psi(t)$; indeed,

$$\begin{aligned}
\psi\big((z_1,t_1)\cdot(z_2,t_2)\big) &= \psi\big(z_1+z_2, t_1+t_2+Q(z_1,z_2)\big) \\
&= \psi(t_1)\psi(t_2)\psi\big(Q(z_1,z_2)\big) \\
&= \psi(z_1,t_1)\psi(z_2,t_2).
\end{aligned}$$

Define $(\rho, S_{\mathcal{Z}})$ to be the induced representation $\mathrm{ind}_{H(\mathcal{Z})}^{H(\mathcal{W})} \psi$ of $H(\mathcal{W})$; that is, ρ acts by right translation on the space

$$\left\{ f \in C^\infty\big(H(\mathcal{W})\big) \;:\; \begin{array}{l} \text{1. } f \text{ has compact support modulo } H(\mathcal{Z}), \\ \text{2. } f(zh) = \psi(z)f(h) \text{ for all } z \in H(\mathcal{Z}), h \in H(\mathcal{W}), \\ \text{3. there exists an open compact subgroup } K_f \text{ of} \\ \quad H(\mathcal{W}) \text{ such that } f(hk) = f(h) \text{ for all } k \in K_f \end{array} \right\}$$

The restriction of ρ to the center of $H(\mathcal{W})$ is given by

$$\big(\rho(0,t)f\big)(h) = f\big(h\cdot(0,t)\big) = f\big((0,t)\cdot h\big) = \psi(t)f(h),$$

hence ρ has central character ψ.

For different such closed subgroups $\mathcal{Z}$ of $\mathcal{W}$, one can build a model for ρ and hence a model for ω. If $\mathcal{Z}$ is a maximal isotropic subspace of $\mathcal{W}$, the model is called Shrödinger's model.

1.7.2. *Schrödinger's Model.* Let $\mathcal{X}+\mathcal{Y}$ be a complete polarization of $\mathcal{W}$. As $\mathcal{X}$ is a maximal isotropic subspace of $\mathcal{W}$, Q is identically zero on $\mathcal{X}$, and hence $\mathcal{X}=\mathcal{X}^{\perp}$. Following the construction above, one obtains the space of functions $S_{\mathcal{X}}$. The space $S_{\mathcal{X}}$ is canonically isomorphic to the space $S(\mathcal{Y})$ of Schwarz functions on $\mathcal{Y}$ via the map $S_{\mathcal{X}} \to S(\mathcal{Y})$ given by $f \mapsto \phi$, where

$$\phi(y) = f(y,0).$$

Proposition 1.10. *In Schrödinger's model, the representation* ρ *takes the form*

$$\begin{aligned}
\big(\rho(x,0)\phi\big)(y_0) &= \psi\big(-2Q(x,y_0)\big)\phi(y_0)\\
\big(\rho(y,0)\phi\big)(y_0) &= \phi(y+y_0)\\
\big(\rho(0,t)\phi\big)(y_0) &= \psi(t)\phi(y_0)
\end{aligned}$$

Proof. Let $t \in F$, $x \in \mathcal{X}$, and $y_0, y \in \mathcal{Y}$. Then

$$\begin{aligned}
\big(\rho(x+y,t)\phi\big)(y_0) &= f\big((y_0,0)\cdot(x+y,t)\big)\\
&= f\big(x+y+y_0\,,\, t+Q(y_0,x+y)\big)\\
&= f\big((x,t+Q(y_0,x)-Q(x,y+y_0))\cdot(y+y_0,0)\big)\\
&= \psi\big(x,t-Q(x,y+2y_0)\big)f(y+y_0,0)\\
&= \psi\big(t-Q(x,y+2y_0)\big)\phi(y+y_0).
\end{aligned}$$

□

The next step is to find the intertwining operator T. The group $\mathrm{Sp}(\mathcal{W})$, under the symplectic basis corresponding to the polarization $\mathcal{X}+\mathcal{Y}$, is generated by the following types of matrices:

$$\begin{aligned}
\underline{x}(b) &= \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \qquad (b \text{ a symmetric } n\times n \text{ matrix}),\\
\underline{h}(b) &= \begin{bmatrix} b & \\ & {}^{\mathsf{T}}b^{-1} \end{bmatrix} \qquad (b \text{ an invertible } n\times n \text{ matrix}),\\
\underline{w} &= \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.
\end{aligned}$$

Proposition 1.11. *The intertwining map* $T : \mathrm{Sp}(\mathcal{W}) \to \mathrm{GL}\big(S(\mathcal{Y})\big)/\mathbb{C}^{\times}$ *is given by*

$$\begin{aligned}
T\big(\underline{x}(b)\big)\phi(y) &= \psi({}^{\mathsf{T}}yby)\phi(y)\\
T\big(\underline{h}(b)\big)\phi(y) &= \phi({}^{\mathsf{T}}by)\\
T(\underline{w})\phi(y) &= \widehat{\phi}(y),
\end{aligned}$$

where $\widehat{\phi}$ *denotes the Fourier transform of* ϕ *with respect to the fixed character* ψ*.*

Proof. Note that the action of $(0,t)$ under ρ will clearly be intertwined by these operators, and hence it remains only to verify the proposition for $\rho_0(x,y) = \rho(x+y,0)$. First, the formula $T(g)\rho = \rho^g T(g)$ is verified for $g = \underline{x}(b)$:

$$\begin{aligned}
&\rho_0^{\underline{x}(b)}(x,y)\Big(T\big(\underline{x}(b)\big)\phi\Big)(y_0)\\
&\qquad= \rho_0(x, bx+y)\Big(T\big(\underline{x}(b)\big)\phi\Big)(y_0)\\
&\qquad= \psi\big(-{}^{\intercal}(x+by)(y+2y_0)\big)\Big(T\big(\underline{x}(b)\big)\phi\Big)(y+y_0)\\
&\qquad= \psi\big(-{}^{\intercal}(x+by)(y+2y_0)\big)\psi\big({}^{\intercal}(y+y_0)b(y+y_0)\big)\phi(y+y_0)\\
&\qquad= \psi({}^{\intercal}y_0by_0)\psi\big(-{}^{\intercal}x(y+2y_0)\big)\phi(y+y_0)\\
&\qquad= \psi({}^{\intercal}y_0by_0)\big(\rho_0(x,y)\phi\big)(y_0)\\
&\qquad= T\big(\underline{x}(b)\big)\big(\rho_0(x,y)\phi\big)(y_0).
\end{aligned}$$

Next, it is verified for $g = \underline{h}(b)$:

$$\begin{aligned}
&\rho_0^{\underline{h}(b)}(x,y)\Big(T\big(\underline{h}(b)\big)\phi\Big)(y_0)\\
&\qquad= \rho_0(bx, {}^{\intercal}b^{-1}y)\Big(T\big(\underline{h}(b)\big)\phi\Big)(y_0)\\
&\qquad= \psi\big(-{}^{\intercal}x(y+2\,{}^{\intercal}by_0)\big)\Big(T\big(\underline{h}(b)\big)\phi\Big)(y_0)\\
&\qquad= \psi\big(-{}^{\intercal}x(y+2\,{}^{\intercal}by_0)\big)\phi(y+{}^{\intercal}by_0)\\
&\qquad= \big(\rho_0(x,y)\phi\big)({}^{\intercal}by_0)\\
&\qquad= T\big(\underline{h}(b)\big)\big(\rho_0(x,y)\phi\big)(y_0).
\end{aligned}$$

Finally, it is verified for $g = \underline{w}$:

$$\begin{aligned}
&\rho_0^{w}(x,y)\big(T(\underline{w})\phi\big)(y_0)\\
&\qquad= \rho_0(y,-x)\widehat{\phi}(y_0)\\
&\qquad= \psi\big(-{}^{\intercal}y(-x+2y_0)\big)\widehat{\phi}(-x+y_0)\\
&\qquad= \psi\big(-{}^{\intercal}y(-x+2y_0)\big)\int_{\mathcal{Y}}\psi\big(2\,{}^{\intercal}u(-x+y_0)\big)\phi(u)du\\
&\qquad= \psi\big(-{}^{\intercal}y(-x+2y_0)\big)\int_{\mathcal{Y}}\psi\big(2\,{}^{\intercal}(v+y)(-x+y_0)\phi(v+y)dv\\
&\qquad= \int_{\mathcal{Y}}\psi(2\,{}^{\intercal}vy_0)\psi\big(-{}^{\intercal}x(y+2v)\big)\phi(v+y)dv\\
&\qquad= \int_{\mathcal{Y}}\psi(2\,{}^{\intercal}vy_0)\big(\rho_0(x,y)\phi\big)(v)dv\\
&\qquad= T(\underline{w})\big(\rho_0(x,y)\phi\big)(y_0).
\end{aligned}$$

∎

Considering $\widetilde{\mathrm{Sp}}(\mathcal{W})$ as a central extension of the symplectic Chevalley group, take $x(b)$, $h(b)$ and w to be the canonical lifts of $\underline{x}(b)$, $\underline{h}(b)$, and $\underline{w}$, as in Section 1.8. The Schrödinger model of the Weil representation ω is given by

$$\begin{aligned} x(b)\phi(y) &= \alpha_b \psi({}^{\intercal}yby)\phi(y) \\ h(b)\phi(y) &= \beta_b |\det(b)|^{1/2}\phi({}^{\intercal}by) \\ w\phi(y) &= \gamma_1 \widehat{\phi}(y) \end{aligned}$$

for some constants $\alpha_b, \beta_b, \gamma_1 \in \mathbb{C}^\times$. The factor $|\det(b)|^{1/2}$ is a normalization term that is included for convenience.

This representation is not irreducible. Since $\psi\big({}^{\intercal}(-y)b(-y)\big) = \psi({}^{\intercal}yby)$, and since the Fourier transform of an even (odd) function is even (odd), the space of even (odd) functions is an ω-invariant subspace of $S(\mathcal{Y})$.

2. Weil Representation of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$

Throughout this section, $\underline{G} = \mathrm{SL}_2(\mathbb{Q}_p)$, $G = \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is a central extension of $\underline{G}$ by $\mathbb{C}^\times$, $\psi = \psi_a$ is an additive character of $\mathbb{Q}_p$ of conductor $c = -\operatorname{val}(a)$, and $V = S(\mathbb{Q}_p)$ is the Schrödinger model of the Weil representation of G associated to ψ. In addition, ϕ_m will denote the characteristic function of $p^m\mathbb{Z}_p$.

The group $\underline{G}$, as a Chevalley group, is generated by the elements

$$\begin{aligned} \underline{x}(t) &= \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}, \qquad \text{where } t \in \mathbb{Q}_p, \\ \underline{y}(t) &= \begin{bmatrix} 1 & \\ t & 1 \end{bmatrix}, \qquad \text{where } t \in \mathbb{Q}_p. \end{aligned}$$

For $t \in \mathbb{Q}_p^\times$, the elements $\underline{w}(t)$ and $\underline{h}(t)$ are defined by

$$\underline{w}(t) = \underline{x}(t)\underline{y}(-t^{-1})\underline{x}(t) = \begin{bmatrix} & t \\ -t^{-1} & \end{bmatrix},$$

$$\underline{h}(t) = \underline{w}(t)\underline{w}(-1) = \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix}.$$

Let $x(t)$ and $y(t)$ be the unique lifts of $\underline{x}(t)$ and $\underline{y}(t)$; these elements generate the central extension G of $\underline{G}$. For $t \in \mathbb{Q}_p^\times$, define the elements $w(t)$ and $h(t)$ of G by

$$w(t) = x(t)y(-1/t)x(t), \qquad \text{and} \qquad h(t) = w(t)w(-1).$$

These are the canonical lifts of $\underline{w}(t)$ and $\underline{h}(t)$, respectively.

The elements $w_0 = w(1/p)$ and $w_1 = w(1)$ form a set of representatives of the generators $\{s_0, s_1\}$ of the affine Weyl group W^{aff}. The full set of representatives of W^{aff} in G is $\big\{w(p^n), h(p^n) : n \in \mathbb{Z}\big\}$.

The Schrödinger model of the Weil representation (ω, V) is given by

$$\begin{aligned} x(t)\phi(y) &= \alpha_t\psi(ty^2)\phi(y) \\ h(t)\phi(y) &= \beta_t|t|^{1/2}\phi(ty) \\ w_1\phi(y) &= \gamma_1\widehat{\phi}(y), \end{aligned}$$

with the Fourier transform $\widehat{\phi}$ of ϕ as in Section 1.6. It will be useful to consider the action of $w(t)$ which, for $\gamma_t = \beta_t\gamma_1$, is given by

$$w(t)\phi(y) = \gamma_t|t|^{1/2}\widehat{\phi}(ty).$$

2.1. **The Constants α_t, β_t, and γ_t.**

Proposition 2.1. *The constant α_t is equal to 1 for all $t \in \mathbb{Q}_p$.*

Proof. In any central extension of $\mathrm{SL}_2(\mathbb{Q}_p)$, one has $h(r)x(s)h(r)^{-1} = x(sr^2)$, and hence

$$\big[h(r), x(s)\big] = x\big(s(r^2-1)\big).$$

Since $\mathbb{Q}_p^\times$ has an element whose square is not equal to 1, then every $x(t)$ can be expressed as a commutator $x(t) = [h(r), x(s)]$ where $t = s(r^2-1)$. Therefore, for any $\phi \in V$,

$$\begin{aligned} \alpha_t\psi(ty^2)\phi(y) &= x(t)\phi(y) \\ &= h(r)x(s)h(r^{-1})x(-s)\phi(y) \\ &= \psi\big(s(r^2-1)y^2\big)\phi(y), \end{aligned}$$

so $\alpha_t = 1$. This fact is independent of the choice of ψ. □

Proposition 2.2. *Let t be an element of $\mathbb{Q}_p^\times$ with $n = \mathrm{val}(t)$. Let m be the integer such that $n-c-1 \leq 2m \leq n-c$, and set $\ell = m + (c-\delta)$ with δ as in Proposition 1.8 so that $\widehat{\phi}_{-m} = \mathrm{vol}(p^{-m}\mathbb{Z}_p)\phi_\ell$. Then*

$$\gamma_t = p^{n/2}\int_{p^\ell\mathbb{Z}_p}\psi(u^2/t)du.$$

Proof. On the one hand,

$$w(t)\phi_{-m}(0) = \gamma_t p^{-n/2}\,\mathrm{vol}(p^{-m}\mathbb{Z}_p)\phi_\ell(0) = \gamma_t p^{-n/2}\,\mathrm{vol}(p^{-m}\mathbb{Z}_p).$$

On the other hand, in G one has

$$w(t) = x(t)y(-1/t)x(t) = x(t)w_1x(1/t)w_1^{-1}x(t),$$

so $w(t)\phi_m(0)$ may be computed using

$$\begin{aligned} w(t)\phi_m(y) &= x(t)w_1x(1/t)w_1^{-1}x(t)\phi_{-m}(y) \\ &= x(t)w_1x(1/t)w_1^{-1}\phi_{-m}(y) \\ &= \gamma_1^{-1}\operatorname{vol}(p^{-m}\mathbb{Z}_p)x(t)w_1x(1/t)\phi_\ell(y) \\ &= \gamma_1^{-1}\operatorname{vol}(p^{-m}\mathbb{Z}_p)x(t)w_1\big(\psi(y^2/t)\phi_\ell(y)\big) \\ &= \operatorname{vol}(p^{-m}\mathbb{Z}_p)\psi(ty^2)\int_{p^\ell\mathbb{Z}_p}\psi(u^2/t)\psi(2uy)du; \end{aligned}$$

that is,

$$w(t)\phi_{-m}(0) = \operatorname{vol}(p^{-m}\mathbb{Z}_p)\int_{p^\ell\mathbb{Z}_p}\psi(u^2/t)du.$$

□

Corollary 2.3. *For any $s \in \mathbb{Q}_p^\times$ let s' in $\mathbb{Z}_p^\times$ be given by $s' = p^{-\operatorname{val}(s)}s$. The constant γ_t is given as follows. For $p = 2$,*

$$\gamma_t = \begin{cases} \psi_1(a't'/8) & \text{if } \operatorname{val}(a), \operatorname{val}(t) \text{ different parity,} \\ \begin{cases} \psi_1(1/8) & \text{if } a' \equiv t' \ (4) \\ \psi_1(7/8) & \text{otherwise} \end{cases} & \text{if } \operatorname{val}(a), \operatorname{val}(t) \text{ same parity.} \end{cases}$$

For $p \neq 2$,

$$\gamma_t = \begin{cases} 1 & \text{if } \operatorname{val}(a), \operatorname{val}(t) \text{ different parity,} \\ \begin{cases} \left(\frac{a't'}{p}\right) & \text{if } p \equiv 1 \ (4) \\ \left(\frac{a't'}{p}\right) i & \text{if } p \equiv 3 \ (4) \end{cases} & \text{if } \operatorname{val}(a), \operatorname{val}(t) \text{ same parity.} \end{cases}$$

Proof. This can be computed explicitly using the previous proposition in the four cases, recalling that $\delta = 1$ for $p = 2$ and $\delta = 0$ for $p \neq 2$.

Case 1 ($p = 2$, $n = 2m + c + 1$): If $u \in 2^\ell x + 2^{\ell+3}\mathbb{Z}_2$, then $u^2 \in 2^{n+c}(x^2/8 + \mathbb{Z}_2)$, and hence $\psi_a(u^2/t) = \psi_{a'/t'}(x^2/8)$, which depends on the values of a' and t' modulo 8. Therefore,

$$\begin{aligned} \gamma_t &= 2^{n/2}\sum_{x\in\mathbb{Z}/8\mathbb{Z}}\int_{2^\ell x+2^{\ell+3}\mathbb{Z}_2}\psi(u^2/t)du \\ &= 2^{n/2}\operatorname{vol}(2^{\ell+3}\mathbb{Z}_2)\sum_{x\in\mathbb{Z}/8\mathbb{Z}}\psi_{a'/t'}(x^2/8) \\ &= 2^{-2}\big(4\psi_{a'/t'}(1/8)\big) \\ &= \psi_{a'/t'}(1/8). \end{aligned}$$

Every invertible element in $\mathbb{Z}/8\mathbb{Z}$ has order 2, so $\psi_{a'/t'}(1/8) = \psi_{a't'}(1/8)$.

Case 2 ($p = 2$, $n = 2m + c$): If $u \in 2^\ell x + 2^{\ell+2}\mathbb{Z}_2$, then $u^2 \in 2^{n+c}(x^2/4 + \mathbb{Z}_2)$, and hence $\psi_a(u^2/t) = \psi_{a'/t'}(x^2/4)$, which depends on the values of a' and t' modulo 4. Therefore,

$$\begin{aligned}
\gamma_t &= 2^{n/2} \sum_{x \in \mathbb{Z}/4\mathbb{Z}} \int_{2^\ell x + 2^{\ell+2}\mathbb{Z}_2} \psi(u^2/t) du \\
&= 2^{n/2} \operatorname{vol}(2^{\ell+2}\mathbb{Z}_2) \sum_{x \in \mathbb{Z}/4\mathbb{Z}} \psi_{a'/t'}(x^2/4) \\
&= 2^{-3/2}\big(2 + 2\psi_{a'/t'}(1/4)\big) \\
&= \frac{1}{\sqrt{2}} + \frac{\psi_{a'/t'}(1/4)}{\sqrt{2}} \\
&= \begin{cases} \psi_1(1/8) & \text{if } a' \equiv t' \ (4) \\ \psi_1(7/8) & \text{otherwise.} \end{cases}
\end{aligned}$$

Case 3 ($p \neq 2$, $n = 2m + c + 1$): If $u \in p^\ell x + p^{\ell+1}\mathbb{Z}_p$, then $u^2 \in p^{n+c}(x^2/p + \mathbb{Z}_p)$, and hence $\psi_a(u^2/t) = \psi_{a'/t'}(x^2/p)$, which depends on the values of a' and t' modulo p. Therefore,

$$\begin{aligned}
\gamma_t &= p^{n/2} \sum_{x \in \mathbb{F}_p} \int_{p^\ell x + p^{\ell+1}\mathbb{Z}_p} \psi(u^2/t) du \\
&= p^{n/2} \operatorname{vol}(p^{\ell+1}\mathbb{Z}_p) \sum_{x \in \mathbb{F}_p} \psi_{a'/t'}(x^2/p) \\
&= p^{-1/2}\Big[\psi_{a'/t'}(0) + 2\sum_{x \in S} \psi_{a'/t'}(x/p) - \sum_{x \in \mathbb{F}_p} \psi_{a'/t'}(x/p)\Big] \\
&= p^{-1/2} \sum_{x \in \mathbb{F}_p^\times} \left(\frac{x}{p}\right) \psi_{a'/t'}(x/p).
\end{aligned}$$

The set S appearing in the calculation above is the set of squares in $\mathbb{F}_p$. The sum in the final line is a quadratic Gauss sum, and, since

$$\sum_{x \in \mathbb{F}_p^\times} \left(\frac{x}{p}\right) \psi_1(x/p) = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \ (4) \\ i\sqrt{p} & \text{if } p \equiv 3 \ (4), \end{cases}$$

a change of variables implies that

$$\gamma_t = \begin{cases} \left(\frac{a'/t'}{p}\right) & \text{if } p \equiv 1 \ (4) \\ \left(\frac{a'/t'}{p}\right) i & \text{if } p \equiv 3 \ (4). \end{cases}$$

The set S of squares in $\mathbb{F}_p$ is closed under inversion, so $\left(\dfrac{a'/t'}{p}\right) = \left(\dfrac{a't'}{p}\right)$.

Case 4 ($p \neq 2$, $n = 2m + c$): If $u \in p^\ell\mathbb{Z}_p$, then $u^2 \in p^{n+c}\mathbb{Z}_p$, and hence $\psi_a(u^2/t) = 1$. Therefore,

$$\gamma_t = p^{n/2}\operatorname{vol}(p^\ell\mathbb{Z}_p) = 1.$$

$\square$

Corollary 2.4. *If $a = p^{-c}$, then γ_1 (the Weil index associated to $\psi = \psi_a$) is given by*

$$\gamma_1 = \begin{cases} \psi_1(1/8) & \text{if } p = 2, \\ i & \text{if } p \equiv 3\ (4) \text{ and } c \text{ is odd}, \\ 1 & \text{otherwise.} \end{cases}$$

Corollary 2.5. *The constant β_t is a 4th root of unity, so G could be taken to be a two-fold central extension of $\underline{G}$.*

Proof. Since $\beta_t = \gamma_t/\gamma_1$, this corollary follows from the fact that γ_t is a primitive 8th root of unity for $p = 2$ and a 4th root of unity for $p \neq 2$. $\square$

2.2. **Iwahori-fixed Vectors.** Let $\underline{I}$ be the Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ given by

$$\underline{I} = \left\{ \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) : a, b, c, d \in \mathbb{Z}_p \right\},$$

and let I be its full inverse image in $G = \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. Then I is generated by the elements

$$\{h(t) : t \in \mathbb{Z}_p^\times\} \cup \{x(t) : t \in \mathbb{Z}_p\} \cup \{y(t) : t \in p\mathbb{Z}_p\}.$$

Suppose that ϕ is an element of the Weil representation V that is supported on $p^m\mathbb{Z}_p$ and constant on $p^n\mathbb{Z}_p$-cosets. For ϕ to be a fixed vector under the action of I, it must be true that $x(t)\phi = \phi$ for all $t \in \mathbb{Z}_p$ and $x(t)\widehat{\phi} = \widehat{\phi}$ for all $t \in p\mathbb{Z}_p$. According to Proposition 1.7, $\widehat{\phi}$ is supported on $p^{-n+(c-\delta)}\mathbb{Z}_p$. Since $x(t)$ acts by $\psi(ty^2)$, ϕ can be fixed under I only if

$$\left\{ \begin{array}{l} \psi(ty^2) = 1 \text{ for } t \in \mathbb{Z}_p, y \in p^m\mathbb{Z}_p \\ \psi(ty^2) = 1 \text{ for } t \in p\mathbb{Z}_p, y \in p^{-n+(c-\delta)}\mathbb{Z}_p \end{array} \right\};$$

that is, only if $c \leq 2m \leq 2n \leq c + 1 - 2\delta$. In particular, there are no Iwahori-fixed vectors for $p = 2$.

2.3. **A Minimal Type for $p = 2$.** For the remainder of the section, the discussion will be restricted to the case $p = 2$ with the additive character $\psi = \psi_{1/2}$ of $\mathbb{Q}_2$ which is trivial on $2\mathbb{Z}_2$. In this setting, the Fourier transform on ϕ_m is given by $\widehat{\phi}_m = 2^{-m}\phi_{-m}$, and the volume of $\mathbb{Z}_2$ is equal to 1.

Let K be the open compact subgroup of $I \subset G$ generated by

$$\{h(t) : t \in \mathbb{Z}_2^\times\} \cup \{x(t) : t \in 2\mathbb{Z}_2\} \cup \{y(t) : t \in 2\mathbb{Z}_2\};$$

that is, K is the full inverse image of the group

$$\underline{K} = \left\{ \begin{bmatrix} a & 2b \\ 2c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}_2) : a, b, c, d \in \mathbb{Z}_2 \right\}.$$

The subspace $\mathbb{C}\phi_0$ of V is K-stable under the Weil representation since

$$\begin{aligned} x(2)\phi_0 &= \phi_0, \\ h(t)\phi_0 &= \beta_t\phi_0 \text{ for } t \in \mathbb{Z}_2^\times, \\ w_1\phi_0 &= \gamma_1\phi_0. \end{aligned}$$

Denote by $\bar{\chi} : K \to \mathbb{C}^\times$ the character satisfying $\omega(k)\phi_0 = \bar{\chi}(k)\phi_0$ for $k \in K$ which acts on the nontrivial subspace

$$V^{K,\bar{\chi}} = \big\{\phi \in V : \omega(k)\phi = \bar{\chi}(k)\phi \text{ for all } k \in K\big\}.$$

Proposition 2.6. *The character $\bar{\chi}$ satisfies the following.*

1. $\bar{\chi}\big(x(t)\big) = \bar{\chi}\big(y(t)\big) = 1$ *for* $t \in 2\mathbb{Z}_2$.
2. $\bar{\chi}\big(x(-1)y(t)x(1)\big) = \bar{\chi}\big(x(1)y(t)x(-1)\big) = \psi(-t/4)$ *for* $t \in 2\mathbb{Z}_2$.
3. $V^{K,\bar{\chi}} = \mathbb{C}\phi_0$.

Proof. Since $\widehat{\phi}_0 = \phi_0$ and since $y(t) = w_1x(-t)w_1^{-1}$, in order to prove the first part it suffices to compute the action of $\bar{\chi}\big(x(t)\big)$ on ϕ_0 given by

$$\bar{\chi}\big(x(t)\big)\phi_0(y) = \psi(ty^2)\phi_0(y).$$

Clearly, this acts as 1 on ϕ_0 if and only if $t \in 2\mathbb{Z}_2$.

By the first part of the proposition, $\bar{\chi}\big(x(\pm 2)\big) = 1$, so

$$\begin{aligned} \bar{\chi}\big(x(-1)y(t)x(1)\big) &= \bar{\chi}\big(x(2)\big)\bar{\chi}\big(x(-1)y(t)x(1)\big)\bar{\chi}\big(x(-2)\big) \\ &= \bar{\chi}\big(x(1)y(t)x(-1)\big). \end{aligned}$$

To prove the second part of the proposition, it then suffices to compute the action of

$$x(1)y(t)x(-1) = x(1)w_1x(-t)w_1^{-1}x(-1)$$

on ϕ_0. The action of $x(-1)$ on ϕ_0 is given by

$$\begin{aligned} x(-1)\phi_0(y) &= \psi(-y^2)\phi_0(y) \\ &= \begin{cases} 1 & \text{if } y \in 2\mathbb{Z}_2 \\ -1 & \text{if } y \in \mathbb{Z}_2 \setminus 2\mathbb{Z}_2 \\ 0 & \text{otherwise} \end{cases} \\ &= \big(-\phi_0 + 2\phi_1\big)(y). \end{aligned}$$

Therefore,

$$\begin{aligned} &x(1)w_1x(-t)w_1^{-1}\big(x(-1)\phi_0\big)(y) \\ &\qquad = x(1)w_1x(-t)w_1^{-1}\big(-\phi_0 + 2\phi_1\big)(y) \\ &\qquad = \gamma_1^{-1}x(1)w_1\big(\psi(-ty^2)(-\phi_0 + \phi_{-1})\big)(y) \\ &\qquad = x(1)\int_{\mathbb{Q}_2}\psi(2uy)\psi(-tu^2)(-\phi_0 + \phi_{-1})(u)du \\ &\qquad = \psi(y^2)\int_{1/2+\mathbb{Z}_2}\psi(2uy)\psi(-tu^2)du, \end{aligned}$$

the last equality following from the fact that $-\phi_0 + \phi_{-1}$ is equal to the characteristic function of $1/2 + \mathbb{Z}_2$. Evaluating this expression at $y = 0$ gives

$$x(1)y(t)x(-1)\phi_0(0) = \int_{1/2+\mathbb{Z}_2} \psi(-tu^2)du.$$

If $u \in 1/2 + \mathbb{Z}_2$, then $u^2 \in 1/4 + \mathbb{Z}_2$, so for $t \in 2\mathbb{Z}_2$, $\psi(-tu^2) = \psi(-t/4)$. Since $\mathrm{vol}(\mathbb{Z}_2) = 1$, one has

$$x(1)y(t)x(-1)\phi_0(0) = \psi(-t/4).$$

To prove the third part, it is enough to show that $V^{K,\bar{\chi}} \subset \mathbb{C}\phi_0$. Let ϕ be an arbitrary element of $V^{K,\bar{\chi}}$ which is supported on $2^m\mathbb{Z}_2$ and constant on $2^n\mathbb{Z}_2$-cosets for $m \leq n$. Then $\widehat{\phi}$ is supported on $2^{-n}\mathbb{Z}_2$. Since $x(2)$ and $y(2)$ act trivially on $V^{K,\bar{\chi}}$, one must have that $\psi(2y^2) = 1$ for all $y \in 2^m\mathbb{Z}_2$ and all $y \in 2^{-n}\mathbb{Z}_2$. Therefore, $0 \leq m \leq n \leq 0$, so $\phi \in \mathbb{C}\phi_0$. □

2.4. **The Associated Hecke Algebra.** The Hecke algebra $\mathcal{H} = \mathcal{H}(G/\!\!/K; \chi)$ for this even type of the Weil representation is

$$\mathcal{H} = \left\{ f \in C_c^\infty(G) : f(k_1 x k_2) = \chi(k_1) f(x) \chi(k_2) \text{ for all } k_i \in K, x \in G \right\}$$

with convolution

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) dx.$$

An element f of $\mathcal{H}$ is determined by its value on a set of representatives of $K\backslash G/K$ or, equivalently, of $\underline{K}\backslash\underline{G}/\underline{K}$. By [5],

$$\underline{G} = \mathrm{SL}_2(\mathbb{Q}_2) = \bigsqcup_{s \in W^{\mathrm{aff}}} \underline{I}\ \underline{w}\ \underline{I}$$

where $\underline{I}$ is the Iwahori subgroup, W^{aff} is the affine Weyl group, and $\underline{w}$ is a representative of s in $\underline{G}$.

It is straight-forward to see that $\underline{K}$ is a normal subgroup of $\underline{I}$ and that $\underline{I}/\underline{K}$ and $\underline{K}\backslash\underline{I}$ are each isomorphic to the group $B(2) = \{\underline{x}(t) : t \in \mathbb{F}_2\}$. A set of double coset representatives of $\underline{K}\backslash\underline{G}/\underline{K}$, and hence of $K\backslash G/K$, must then be contained in the set $B(2)\ W^{\mathrm{aff}}\ B(2)$. However, some of these representatives are redundant and will now be eliminated. Working in the central extension G, since $h(t)x(1)h(t)^{-1} = x(t^2)$,

$$Kx(1)h(2^n)K = Kh(2^n)x(2^{-2n})K = Kh(2^n)K \text{ for } n < 0,$$
$$Kh(2^n)x(1)K = Kx(2^{2n})h(2^n)K = Kh(2^n)K \text{ for } n > 0,$$

and

$$Kx(1)h(2^n)x(1)K = \begin{cases} Kx(1)h(2^n)K & \text{if } n > 0, \\ Kh(2^n)x(1)K & \text{if } n < 0, \\ Kh(1)K & \text{if } n = 0. \end{cases}$$

Similarly, since $w(t)x(1)w(t)^{-1} = y(-t^2)$,

$$Kx(1)w(2^n)K = Kw(2^n)y(-2^{-2n})K = Kw(2^n)K \text{ for } n < 0, \text{ and}$$
$$Kw(2^n)x(1)K = Ky(-2^{-2n})w(2^n)K = Kw(2^n)K \text{ for } n < 0.$$

A complete set of double coset representatives for $K\backslash G/K$ is given by

$$\begin{array}{rcl} w(2^n) & : & n \in \mathbb{Z}, \\ h(2^n) & : & n \in \mathbb{Z}, \\ w(2^n)x(1) & : & n \geq 0, \\ x(1)w(2^n) & : & n \geq 0, \\ x(1)w(2^n)x(1) & : & n \geq 0, \\ x(1)h(2^n) & : & n \geq 0, \\ h(2^n)x(1) & : & n < 0. \end{array}$$

Lemma 2.7. *The support of $\mathcal{H}$ is contained in $KW^{\text{aff}}K$.*

Proof. Let f be an element of $\mathcal{H}$. Recalling two of the Steinberg relations, $w(a)x(t) = y(-a^{-2}t)w(a)$ and $h(a)y(t) = y(a^{-2}t)h(a)$, and Proposition 2.6, if $n \geq 0$, then

$$\begin{aligned} f\big(w(2^n)x(1)\big) &= \chi\big(x(2^{2n+1})\big) f\big(w(2^n)x(1)\big) \\ &= f\big(x(2^{2n+1})w(2^n)x(1)\big) \\ &= f\big(w(2^n)y(2)x(1)\big) \\ &= f\big(w(2^n)x(1)\big)\chi\big(x(-1)y(2)x(1)\big) \\ &= if\big(w(2^n)x(1)\big), \end{aligned}$$

$$\begin{aligned} f\big(x(1)w(2^n)\big) &= f\big(x(1)w(2^n)x(-2^{2n+1})\big) \\ &= \chi\big(x(1)y(2)x(-1)\big) f\big(x(1)w(2^n)\big) \\ &= if\big(x(1)w(2^n)\big), \end{aligned}$$

$$\begin{aligned} f\big(x(1)w(2^n)x(1)\big) &= f\big(x(1)w(2^n)x(1)x(-2^{2n+1})\big) \\ &= \chi\big(x(1)y(2)x(-1)\big) f\big(x(1)w(2^n)x(1)\big) \\ &= if\big(x(1)w(2^n)x(1)\big), \end{aligned}$$

$$\begin{aligned} f\big(x(1)h(2^n)\big) &= f\big(x(1)h(2^n)y(2^{2n+1})\big) \\ &= \chi\big(x(1)y(2)x(-1)\big) f\big(x(1)h(2^n)\big) \\ &= if\big(x(1)(2^n)\big), \end{aligned}$$

and, if $n < 0$, then

$$\begin{aligned} f\big(h(2^n)x(1)\big) &= f\big(y(2^{-2n+1})h(2^n)x(1)\big) \\ &= f\big(h(2^n)x(1)\big)\chi\big(x(-1)y(2)x(1)\big) \\ &= if\big(h(2^n)x(1)\big). \end{aligned}$$

Therefore, $f = 0$ on all double cosets besides $Kh(2^n)K$ and $Kw(2^n)K$. $\square$

In order to study the structure of $\mathcal{H}$, two subalgebras corresponding to the representatives w_0 and w_1 of the two generators of the affine Weyl group are introduced. For $i = 0, 1$, define

1. P_i to be the group generated by K and w_i,
2. V_i to be the subspace $P_i\phi_0$ of V,
3. U_i to be the induced representation $\mathrm{ind}_K^{P_i}\chi$,
4. $\mathcal{H}_i$ to be the Hecke subalgebra $\mathcal{H}(P_i/\!\!/K;\chi)$.

Lemma 2.8. *V_0 is 2-dimensional and V_1 is 1-dimensional.*

Proof. It was already seen that the space $\mathbb{C}\phi_0$ is K-stable; in addition, $w_1\phi_0 \in \mathbb{C}\phi_0$, so

$$V_1 = P_1\phi_0 = \mathbb{C}\phi_0.$$

It will now be shown that $V_0 = \mathbb{C}\phi_0 \oplus \mathbb{C}\phi_1$. The element $w_0 = w(1/2)$ acts on ϕ_0 and ϕ_1 by

$$w(1/2)\phi_0(y) \in \mathbb{C}\widehat{\phi}_0(y/2) = \mathbb{C}\phi_0(y/2) = \mathbb{C}\phi_1(y),$$
$$w(1/2)\phi_1(y) \in \mathbb{C}\widehat{\phi}_1(y/2) = \mathbb{C}\phi_{-1}(y/2) = \mathbb{C}\phi_0(y).$$

Since $y(2) = w_1x(-2)w_1^{-1}$ and $x(-2)\phi_{-1} \in \mathbb{C}\phi_0 \oplus \mathbb{C}\phi_{-1}$, one has

$$y(2)\phi_1 = w_1x(-2)w_1^{-1}\phi_1 = \frac{1}{2}\gamma_1^{-1}w_1x(-2)\phi_{-1} \in \mathbb{C}\phi_0 \oplus \mathbb{C}\phi_1.$$

Hence, $K\phi_0$, $w_0\phi_0$, $K\phi_1$, and $w_0\phi_1$ are all in $\mathbb{C}\phi_0 \oplus \mathbb{C}\phi_1$. □

Lemma 2.9. *U_0 is 6-dimensional and U_1 is 2-dimensional.*

Proof. The dimension of U_i is equal to the index of K in P_i. In the $i = 1$ case, the element w_1 normalizes K, giving that $P_1 = K \cup Kw_1$, so $\dim U_1 = 2$. For the $i = 0$ case, the group P_0 is exactly the group K_{α_0} from Section 1.3.2. Under the isomorphism e_{α_0}, P_0 is isomorphic to $\mathrm{SL}_2(\mathbb{Z}_2)$ and K is isomorphic to the group

$$\left\{ \begin{bmatrix} a & b \\ 4c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}_2) : a, b, c, d \in \mathbb{Z}_2 \right\}.$$

Therefore, the index of K in P_0 is equal to the index of the first congruence subgroup $\Gamma_0(4)$ in $\mathrm{SL}_2(\mathbb{Z})$. This, in turn, is equal to the index of the subgroup of upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ has 48 elements and 8 of them are upper triangular, the dimension of U_0 is equal to 6. □

Lemma 2.10. *Both $\mathcal{H}_0$ and $\mathcal{H}_1$ are 2-dimensional.*

Proof. The group P_i is contained in the parahoric subgroup $I \cup Iw_iI$, which intersects W^{aff} at 1 and w_i. Therefore, by Lemma 2.7, any element of $\mathcal{H}_i$ can only be supported on K and Kw_iK, so $\mathcal{H}_i$ is at most 2-dimensional. On the other hand, the characteristic function of K is the identity element of $\mathcal{H}$, and hence of $\mathcal{H}_i$, so $\mathcal{H}_i$ is at least 1-dimensional. By Proposition 1.3,

$$\mathrm{Hom}_{P_i}(V_i^*, U_i) = \mathrm{Hom}_K(V_i^*, \chi) = \mathrm{Hom}_K(\bar{\chi}, V_i) = V_i^{K,\bar{\chi}} = \mathbb{C}\phi_0,$$

and so there exists a P_i-homomorphism from V_i^* to U_i. For both $i = 0$ and $i = 1$, the dimension of V_i is less than the dimension of U_i, hence U_i must be reducible. Therefore, $\mathrm{End}(U_i)$, which by Proposition 1.5 is isomorphic to $\mathcal{H}_i$, cannot be 1-dimensional. □

Lemma 2.11. *There exist elements T_0 of $\mathcal{H}_0$ and T_1 of $\mathcal{H}_1$ which satisfy the quadratic relations*

$$(T_0 - 2)(T_0 + 1) = 0 \qquad \textit{and} \qquad (T_1 - 1)(T_1 + 1) = 0.$$

Proof. Let T_i be the element of $\mathcal{H}_i$ supported on Kw_iK normalized as follows. Since $\mathcal{H}_i$ is 2-dimensional, T_i acts as an endomorphism on U_i with two eigenvalues λ_i and μ_i corresponding to eigenspaces of dimensions m_i and n_i. By Proposition 1.5, since T_i is not supported on K, T_i acts as a trace-zero endomorphism on U_i, giving that

$$\lambda_i m_i + \mu_i n_i = 0.$$

By Frobenius reciprocity, V_i is one of the eigenspaces, say that of dimension m_i. The element T_i is normalized to act by $\mu_i = -1$ on the eigenspace of dimension n_i (the one not containing ϕ_0). The other eigenvalue then is given by

$$\lambda_i = \frac{n_i}{m_i} = \begin{cases} 4/2 = 2 & \text{if } i = 0, \\ 1/1 = 1 & \text{if } i = 1. \end{cases}$$

□

Lemma 2.12. *The support of $\mathcal{H}$ is $KW^{\mathrm{aff}}K$.*

Proof. From Lemma 2.7, it is enough to construct nonzero elements of $\mathcal{H}$ supported on each K-double coset of W^{aff}.

Let w be a representative of an element s of W^{aff}, and write a minimal expression for s, say as the product $s = s_{i_1} \dots s_{i_m}$ of simple reflections s_0 and s_1. Define the element T_w of $\mathcal{H}$ by $T_w = T_{i_1} \cdots T_{i_m}$. The minimal expression for s is unique since there are no braid relations in W^{aff}, and hence T_w is well-defined. The Hecke algebra $\mathcal{H}$ has a one-dimensional representation $\mathbb{C}\phi_0$ on which T_0 and T_1 act by the scalars 2 and 1, respectively. In particular, T_w acts on ϕ_0 as a product of 2's and 1's, so $T_w \neq 0$. Since w_0 and w_1 are representatives of s_0 and s_1 in G, the support of T_w satisfies

$$\mathrm{supp}(T_w) = Kw_{i_1}Kw_{i_2}K \cdots Kw_{i_m}K \subset Iw_{i_1}Iw_{i_2}I \cdots Iw_{i_m}I = IwI,$$

the last equality holding since the expression of s in terms of simple reflections is minimal. The double coset IwI is the union of finitely many K-double cosets; however, elements of $\mathcal{H}$ are not supported outside of those K-double cosets parametrized by W^{aff}. Therefore, the support of T_w must be equal to KwK. Moreover, the set of nonzero elements $\{T_w : w \in W^{\mathrm{aff}}\}$ forms a basis for $\mathcal{H}$. □

Theorem 2.13. *Abstractly, $\mathcal{H}$ is the algebra generated by T_0 and T_1 subject to the quadratic relations $(T_0 - 2)(T_0 + 1) = 0$ and $(T_1 - 1)(T_1 + 1) = 0$. In particular, $\mathcal{H}$ is isomorphic to the Iwahori-Hecke algebra of* $\mathrm{PGL}_2(\mathbb{Q}_2)$.

Proof. Let A be the Iwahori-Hecke algebra of $\mathrm{PGL}_2(\mathbb{Q}_2)$. Then A is generated by two elements t_0, t_1 subject only to the quadratic relations $(t_0 - 2)(t_0+1) = 0$ and $(t_1 - 1)(t_1+1) = 0$. For w a representative of s in W^{aff}, write $s = s_{i_1} \dots s_{i_m}$ as a reduced expression, and define the element t_w of A by $t_w = t_{i_1} \dots t_{i_m}$. These elements t_w form a basis for A (see [5]). Define a map $A \to \mathcal{H}$ by $t_w \mapsto T_w$. Since $\mathcal{H}$ satisfies the same relations as A, it is a homomorphism that sends a basis to a basis and is thus an isomorphism. $\square$

3. Weil Representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{Q}_2)$

Throughout this section, $\mathcal{W}$ is a $2n$-dimensional symplectic vector space over $\mathbb{Q}_2$, $\mathcal{X}+\mathcal{Y}$ is a complete polarization of $\mathcal{W}$, $\underline{G} = \mathrm{Sp}(\mathcal{W})$ is the symplectic group, $G = \widetilde{\mathrm{Sp}}(\mathcal{W})$ is a central extension of $\underline{G}$ by $\mathbb{C}^\times$, $\psi = \psi_{1/2}$ is an additive character of $\mathbb{Q}_2$ of conductor $c = 1$, and $V = S(\mathcal{Y}) = S(\mathbb{Q}_2^n)$ is the Schrödinger model of the Weil representation of G with respect to ψ. Recall that the elements $x(b)$, $h(b)$, and w in G are lifts of

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} b & 0 \\ 0 & {}^{\intercal}b^{-1} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{respectively,}$$

and that the Weil representation (ω, V) is given by

$$\begin{aligned} x(b)\phi(y) &= \psi({}^{\intercal}yby)\phi(y) \\ h(b)\phi(y) &= \beta_b |\det(b)|^{1/2} \phi({}^{\intercal}by) \\ w\phi(y) &= \gamma_1 \widehat{\phi}(y). \end{aligned}$$

Since $x(b)$ can be expressed as a product of elements in the positive root groups, the usual constant α_b is the product of some of the α_t from the $\widetilde{\mathrm{SL}}_2$ case, each of which is equal to 1. Similarly, if b is diagonal, then $h(b)$ is the product of some of the $h_\alpha(t)$, so the constant β_b is a 4th root of unity. In particular, the central extension $G = \widetilde{\mathrm{Sp}}(\mathcal{W})$ could be taken to be a two-fold central extension.

As a central extension of a Chevalley group, G is generated by $x_\alpha(t)$ for $\alpha \in \Phi$ and $t \in \mathbb{Q}_2$, and is subject to Steinberg's relations for covering groups (see Section 1.3.3). Using the notation of Section 1.2, for a positive root α, X_α is one of

$$\begin{bmatrix} E_{ji} & 0 \\ 0 & -E_{ij} \end{bmatrix}, \quad \begin{bmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & E_{ii} \\ 0 & 0 \end{bmatrix}.$$

In the first case, $x_\alpha(t) = h(b)$ with $b = 1 + tX_\alpha$; in the second and third case, $x_\alpha(t) = x(b)$ with $b = tX_\alpha$. The root α is a short root in the first two cases and a long root in the third case.

Define the elements $w_0, w_1, \dots, w_n$ of G using the same formulas as in the linear case (see Section 1.3.2). These elements are representatives in G of the simple generators $s_0, s_1, \dots, s_n$ of the affine Weyl group W^{aff}.

3.1. **A Minimal Type for $p = 2$.** The Iwahori subgroup I of G is the full inverse image of the Iwahori subgroup $\underline{I}$ of $\underline{G}$, so that I is generated by

$$\left\{h_\alpha(t) : t \in \mathbb{Z}_2^\times\right\} \cup \left\{x_\alpha(t) : \begin{array}{c} t \in \mathbb{Z}_2 \text{ for } \alpha > 0 \\ t \in 2\mathbb{Z}_2 \text{ for } \alpha < 0 \end{array}\right\}.$$

For α a postive long root, the action of $x_\alpha(t)$ (on its particular coordinate) mirrors that of the action of $x(t)$ in the $\widetilde{\mathrm{SL}}_2$ setting. By looking at each of the long roots, one sees that the Weil representation can only have Iwahori-fixed vectors if $p \neq 2$, in which case the fixed vector must be a multiple of the characteristic function of the standard lattice $\mathbb{Z}_p^n$. This, along with the preceding section, motivates the definition (in the $p = 2$ setting) of the subgroup K of I generated by

$$\left\{h_\alpha(t) : t \in \mathbb{Z}_2^\times\right\} \cup \left\{x_\alpha(t) : \begin{array}{c} t \in \mathbb{Z}_2 \text{ for } \alpha > 0, \text{ short} \\ t \in 2\mathbb{Z}_2 \text{ for } \alpha > 0, \text{ long} \\ t \in 2\mathbb{Z}_2 \text{ for } \alpha < 0 \end{array}\right\}.$$

A very nice way to think of K is the following. Recall the general definition of the Iwahori subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$ as the inverse image (under projection modulo 2) of the Borel subgroup B of the finite symplectic group $\mathrm{Sp}_{2n}(2)$. The finite orthogonal group $\mathrm{O}_{2n}(2)$ is defined as the set of linear operators of a $2n$-dimensional vector space over $\mathbb{F}_2$ under which a symmetric quadratic form q is invariant. Associated with q is the bilinear form (x, y) given by $(x, y) = q(x + y) + q(x) + q(y)$, which is clearly symmetric. However, in characteristic 2, a symmetric form is also skew-symmetric, so $\mathrm{O}_{2n}(2)$ is a subgroup of $\mathrm{Sp}_{2n}(2)$. As Chevalley groups, the finite orthogonal group $\mathrm{O}_{2n}(2)$ is precisely realized as the subgroup generated by the short root groups (see [1]). Under this identification, the Borel subgroup B' of $\mathrm{O}_{2n}(2)$ is the subgroup of B generated by the short positive roots, $\underline{K}$ is the pull-back of B' to $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$ and K is the full inverse image of $\underline{K}$ in the central extension G.

$$\begin{tikzcd}
\underline{K} \arrow[r] \arrow[d] & \underline{I} \arrow[r] \arrow[d] & \mathrm{Sp}_{2n}(\mathbb{Z}_2) \arrow[d] \\
B' \arrow[r] & B \arrow[r] & \mathrm{Sp}_{2n}(2)
\end{tikzcd}$$

Lemma 3.1. *Let $\phi_0 \in V$ be the characterstic function of the standard lattice $\mathbb{Z}_2^n$. Then $\mathbb{C}\phi_0$ is a K-stable subspace of V.*

Proof. It must be shown that the generators of K preserve the space $\mathbb{C}\phi_0$. In the four cases that follow, let $y = (y_1, \ldots, y_n)$ be an element of $\mathbb{Z}_2^n$.

1. A generator $h_\alpha(t)$ is of the form $h(b)$ for b a diagonal matrix with entries in $\mathbb{Z}_2^\times$. Then ${}^\intercal b y$ is in $\mathbb{Z}_2^n$, so $h_\alpha(t)\phi_0 \in \mathbb{C}\phi_0$.
2. Let $t \in \mathbb{Z}_2$ and α a short root so that $x_\alpha(t)$ is of the form $h(b)$ with $b = 1 + tE_{ij}$. Then ${}^\intercal b$ acts on y by replacing y_j with $y_j + ty_i$, which is still in $\mathbb{Z}_2^n$, Therefore, $x_\alpha(t)\phi_0 \in \mathbb{C}\phi_0$.

3. Let $t \in \mathbb{Z}_2$ and α a short root so that $x_\alpha(t)$ is of the form $x(b)$ with $b = t(E_{ij} + E_{ji})$. Then ${}^\intercal yby = 2ty_iy_j \in 2\mathbb{Z}_2$, and hence $x_\alpha(t)\phi_0 = \phi_0$.
4. Let $t \in 2\mathbb{Z}_2$ and α a long root. Then $x_\alpha(t)$ is of the form $x(b)$ with $b = tE_{ii}$, and ${}^\intercal yby = ty_i^2 \in 2\mathbb{Z}_2$. Therefore, $x_\alpha(t)\phi_0 = \phi_0$.

□

Proposition 3.2. *Let $\bar{\chi}$ be the character of K defined by $\omega(k)\phi_0 = \bar{\chi}(k)\phi_0$, which acts on the nontrivial subspace*

$$V^{K,\bar{\chi}} = \{\phi \in V : \omega(k)\phi = \bar{\chi}(k)\phi \text{ for all } k \in K\}.$$

Then

1. $\bar{\chi}\big(x_\alpha(t)\big) = 1$, *for $t \in 2\mathbb{Z}_2$ and α postive, long.*
2. $\bar{\chi}\big(x_\alpha(-1)x_{-\alpha}(t)x_\alpha(1)\big) = \psi(-t/4)$, *for $t \in 2\mathbb{Z}_2$ and α positive, long.*
3. $V^{K,\chi} = \mathbb{C}\phi_0$.

Proof. The first part follows from the proof of the previous lemma. The second part follows from Lemma 2.6 since the action of $x_\alpha(-1)x_{-\alpha}(t)x_\alpha(1)$ on the first component of $S(\mathcal{Y})$ is the same as the $\widetilde{\mathrm{SL}}_2$ action of $x(-1)y(t)x(1)$ from before. For the third part, since a function ϕ in V is the tensor product of functions on $\mathbb{Q}_2$, each piece of a function $\phi \in V^{K,\bar{\chi}}$ must be be supported on $\mathbb{Z}_2$ and trivial on $\mathbb{Z}_2$-cosets, and hence must be a multiple of ϕ_0. □

3.2. **The Associated Hecke Algebra.** The Hecke algebra $\mathcal{H} = \mathcal{H}(G/\!\!/K; \chi)$ for this even type of the Weil representation is

$$\mathcal{H} = \{f \in C_c^\infty : f(k_1xk_2) = \chi(k_1)f(x)\chi(k_2) \text{ for all } k_i \in K, x \in G\}.$$

Lemma 3.3. *The support of $\mathcal{H}$ is contained in $KW^{\mathrm{aff}}K$.*

Proof. As before, an element f of $\mathcal{H}$ is determined by its value on a set of representatives of $K\backslash G/K$ or, equivalently, of $\underline{K}\backslash\underline{G}/\underline{K}$. By [5], the $\underline{I}$-double cosets are parametrized by W^{aff}. In addition, the I/K- and $K\backslash I$-cosets have as representatives products of elements of the form $x_\alpha(1)$ for α positive and long. Therefore, a typical double coset of $K\backslash G/K$ is of the form Kx_1wx_2K where w is a representative an element of the affine Weyl group and x_1, x_2 are products of some of those $x_\alpha(1)$'s.

Suppose that $x_\alpha(1)$ occurs in x_1 and that $w^{-1}x_\alpha(1)w = x_\beta(t)$ for some root β and some $t \in 2\mathbb{Z}_2$. All of the long root groups commute with each other, so it may be assumed that $x_1 = x_1'x_\alpha(1)$. If $x_{-\beta}(1)$ does not occur in x_2, then

$$Kx_1'x_\alpha(1)wx_2K = Kx_1'wx_2x_\beta(t)K = Kx_1'wx_2K.$$

If $x_{-\beta}(1)$ does occur in x_2, say $x_2 = x_2'x_{-\beta}(1)$, then

$$\begin{aligned} Kx_1'x_\alpha(1)wx_2K &= Kx_1'wx_2'x_\beta(t)x_{-\beta}(1)K \\ &= Kx_1'wx_2'x_{-\beta}(1)\big(x_{-\beta}(-1)x_\beta(t)x_{-\beta}(1)\big)K \\ &= Kx_1'wx_2K, \end{aligned}$$

with the last equality following from the fact that the subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$ consisting of matrices of the form $\begin{bmatrix} a & 2b \\ 2c & d \end{bmatrix}$ is a normal subgroup of the Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$. In either case, if $t \in 2\mathbb{Z}_2$, then $x_\alpha(1)$ may be moved across and absorbed into K.

Supposing now that $x_\alpha(1)$ occurs in x_1 with $w^{-1}x_\alpha(1)w = x_\beta(t)$ for $t \notin 2\mathbb{Z}_2$, suppose further that $x_\beta(1)$ occurs in x_2. Since $wx_\beta(1)w^{-1} = x_\alpha(1/t)$, a similar argument to that above permits the assumption that $1/t \notin 2\mathbb{Z}_2$; that is, it may be assumed that $t \in \mathbb{Z}_2^\times$, or that $1+t \in 2\mathbb{Z}_2$. Writing $x_1 = x_1' x_\alpha(1)$ and $x_2 = x_2' x_\beta(1)$, one has

$$Kx_1'x_\alpha(1)wx_2'x_\beta(1)K = Kx_1'wx_2'x_\beta(t+1)K = Kx_1'wx_2'K.$$

Therefore, it may be assumed that such an $x_\alpha(1)$ occurring in x_1 precludes the appearance of the corresponding $x_\beta(1)$ in x_2.

In summary, if $x_\alpha(1)$ occurs in x_1 with $w^{-1}x_\alpha(1)w = x_\beta(t)$, it may be assumed that $t \notin 2\mathbb{Z}_2$ and that $x_\beta(1)$ does not occur in x_2. For $f \in \mathcal{H}$, writing $x_1 = x_\alpha(1)x_1'$, one has

$$\begin{aligned}
f(x_1wx_2) &= f(x_1wx_2)\chi\big(x_{-\beta}(2/t)\big) \\
&= f\big(x_1wx_2x_{-\beta}(2/t)\big) \\
&= f\big(x_1wx_{-\beta}(2/t)x_2\big) \\
&= f\big(x_1x_{-\alpha}(2)wx_2\big) \\
&= f\big(x_\alpha(1)x_{-\alpha}(2)x_\alpha(-1)x_1wx_2\big) \\
&= \chi\big(x_\alpha(1)x_{-\alpha}(2)x_\alpha(-1)\big)f(x_1wx_2) \\
&= if(x_1wx_2),
\end{aligned}$$

giving that $f(x_1wx_2) = 0$. Therefore, $\mathcal{H}$ is not supported on any coset Kx_1wx_2K when x_1 is nontrivial. A symmetrical calculation shows that $\mathcal{H}$ is not supported on any coset Kwx_2K when x_2 is nontrivial. Hence the support of $\mathcal{H}$ is at most the set of double cosets parametrized by the affine Weyl group. □

As in the $\widetilde{\mathrm{SL}}_2$ setting, in order to study the structure of $\mathcal{H}$, subalgebras $\mathcal{H}_0, \dots, \mathcal{H}_n$ are introduced. For $i = 0, \dots, n$ define

1. P_i to be the group generated by K and w_i,
2. V_i to be the subspace $P_i\phi_0$ of V,
3. U_i to be the induced representation $\mathrm{ind}_K^{P_i}\chi$,
4. $\mathcal{H}_i$ to be the Hecke subalgebra $\mathcal{H}(P_i /\!/ K; \chi)$.

Lemma 3.4. *The dimension of V_i is*

$$\begin{cases} 2 & \text{if} \quad i = 0, \\ 1 & \text{if} \quad i = 1, \dots, n. \end{cases}$$

Proof. The action of $w_0 = w_{\alpha_*}(1/2) = w_{2\lambda_1}(1/2)$ on the first component of $S(\mathcal{Y})$ is the same as the action of $w(1/2)$ in the $\widetilde{\mathrm{SL}}_2$ case. Similarly, the

action of $w_n = w_{\alpha_n}(1) = w_{2\lambda_n}(1)$ on the last component of $S(\mathcal{Y})$ is the same as the action of $w(1)$ in the $\widetilde{\mathrm{SL}}_2$ case. Therefore, by Lemma 2.8, the dimension of V_0 is 2, and the dimension of V_n is 1. For $i = 1, \ldots, n-1$, w_i is of the form $h(b)$, hence P_i fixes $\mathbb{C}\phi_0$, and the dimension of V_i is 1. $\square$

Lemma 3.5. *The dimension of U_i is*

$$\begin{cases} 6 & \text{if} \quad i = 0, \\ 3 & \text{if} \quad i = 1, \ldots, n-1, \\ 2 & \text{if} \quad i = n. \end{cases}$$

Proof. The dimension of U_i is equal to the index of K in P_i. In the $i = n$ case, the element w_n normalizes K, giving that $Kw_nK = K \cup Kw_n$, so the index of K in P_n is 2. In the other cases, using the notation of Section 1.3.2, the group P_i is exactly the group $K_{\alpha_i}K$, so the index of K in P_i is equal to the index of $K \cap K_{\alpha_i}$ in K_{α_i}. This index will be studied via the map $e_{\alpha_i} : \mathrm{SL}_2(\mathbb{Q}_2) \to G$, under which K_{α_i} is isomorphic to $\mathrm{SL}_2(\mathbb{Z}_2)$. For $i = 0$, the group $K \cap K_{\alpha_0}$ is generated by $x_{\alpha_0}(t)$ and $x_{-\alpha_0}(4t)$ for $t \in \mathbb{Z}_2$. Therefore, $K \cap K_{\alpha_0}$ is isomorphic to the first congruence subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$, which is a subgroup of index 6 (see the proof of Lemma 2.9). For $i = 1, \ldots, n-1$, the group $K \cap K_{\alpha_i}$ is generated by $x_{\alpha_i}(t)$ and $x_{-\alpha_i}(2t)$ for $t \in \mathbb{Z}_2$. Therefore, $K \cap K_{\alpha_i}$ is isomorphic to the Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$, which is a subgroup of index 3. $\square$

Lemma 3.6. *Each $\mathcal{H}_i$ is 2-dimensional.*

Proof. The group P_i is contained in the parahoric subgroup $I \cup Iw_iI$, which intersects W^{aff} only at 1 and w_i. By Lemma 3.3, the only double cosets on which $\mathcal{H}_i$ is supported are K and Kw_iK, so $\mathcal{H}_i$ is at most 2-dimensional. The characteristic function on K is the identity element of each $\mathcal{H}_i$, so $\mathcal{H}_i$ is at least 1-dimensional. As in the proof of Lemma 2.10,

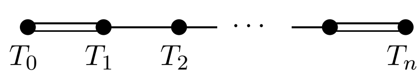
$$\mathrm{Hom}_{P_i}(V_i^*, U_i) = \mathrm{Hom}_K(V_i^*, \chi) = \mathrm{Hom}_K(\bar{\chi}, V_i) = V_i^{K, \bar{\chi}}.$$

Since $\dim V_i \leq \dim U_i$, the induced representation U_i must be reducible, and hence, $\mathrm{End}(U_i) = \mathcal{H}_i$ cannot be 1-dimensional. $\square$

Lemma 3.7. *For $i = 0, \ldots, n$, there exists elements T_i of $\mathcal{H}$, supported on Kw_iK, which satisfy the quadratic relations*

$$\begin{cases} (T_i + 1)(T_i - 2) = 0 & \text{for } i = 0, 1, \ldots, n-1 \\ (T_n + 1)(T_n - 1) = 0 \end{cases}$$

and the braid relations given by the following Coxeter diagram.



Proof. As $\mathcal{H}_i$ is 2-dimensional, an element with support Kw_iK acts as a trace-zero endomorphism of U_i by two eigenvalues λ_i and μ_i with corresponding eigenspaces of dimension m_i and n_i. Let V_i be the eigenspace of

dimension m_i. Choose such an element T_i normalized to act by $\mu_i = -1$ on the other eigenspace (that which does not contain ϕ_0). Since

$$\lambda_i m_i + \mu_i n_i = 0,$$

T_i acts on V_i by

$$\lambda_i = \frac{n_i}{m_i} = \begin{cases} 4/2 = 2 & \text{if } i = 0, \\ 2/1 = 2 & \text{if } i = 1, \ldots, n-1, \\ 1/1 = 1 & \text{if } i = n, \end{cases}$$

which gives the desired quadratic relations.

Let $s_{i_1} s_{i_2} s_{i_1} \cdots = s_{i_2} s_{i_1} s_{i_2} \cdots$ be a braid relation in the affine Weyl group, and let w_{i_1}, w_{i_2} be the corresponding representatives of s_{i_1}, s_{i_2} in the group G. The support of $T_{i_1} T_{i_2} T_{i_1} \cdots$ satisfies

$$K w_{i_1} K w_{i_2} K w_{i_1} K \cdots \subset I w_{i_1} I w_{i_2} I w_{i_1} I \cdots = I w_{i_1} w_{i_2} w_{i_1} \cdots I,$$

while the support of $T_{i_2} T_{i_1} T_{i_2} \cdots$ satisfies

$$K w_{i_2} K w_{i_1} K w_{i_2} K \cdots \subset I w_{i_2} I w_{i_1} I w_{i_2} I \cdots = I w_{i_2} w_{i_1} w_{i_2} \cdots I.$$

The last equalities in these two statements follow from the fact that the corresponding expressions in the affine Weyl group are minimal expressions. In addition, the braid relation of W^{aff} implies that these two I-double cosets are equal. However, the elements of $\mathcal{H}$ are only supported on K-double cosets of W^{aff}, so the two functions $T_{i_1} T_{i_2} T_{i_1} \cdots$ and $T_{i_2} T_{i_1} T_{i_2} \cdots$ must be supported on

$$K w_{i_1} w_{i_2} w_{i_1} \cdots K = K w_{i_2} w_{i_1} w_{i_2} \cdots K.$$

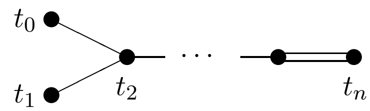
It follows that they must be equal up to multiplication by a scalar. By considering the representation $\mathbb{C}\phi_0$ of $\mathcal{H}$ on which T_{i_k} acts by either 1 or 2, one concludes that the functions must, in fact, be equal. Therefore, every braid relation of W^{aff} yields a corresponding braid relation of $\mathcal{H}$. $\square$

Lemma 3.8. *The support of $\mathcal{H}$ is $KW^{\text{aff}}K$.*

Proof. From Lemma 3.3, it remains only to verify that $\mathcal{H}$ contains a nontrivial element supported on every K-double coset of W^{afff}. Let w be a representative of an element s of the affine Weyl group, and write a minimal expression $s = s_{i_1} \ldots s_{i_r}$ for s as the product of simple reflections. Define the element $T_w = T_{i_1} \ldots T_{i_r}$ of $\mathcal{H}$. The Hecke algebra satisfies the same braid relations as the affine Weyl group, so this definition of T_w is independent of the choice of minimal expression. As in the proof of the previous lemma, this element is supported on KwK. Since each T_{i_k} acts on ϕ_0 by 1 or 2, $T_w \neq 0$. Moreover, the set of nonzero elements $\{T_w : w \in W^{\text{aff}}\}$ forms a basis for $\mathcal{H}$. $\square$

Theorem 3.9. *Abstractly, $\mathcal{H}$ is the algebra generated by $T_0, \ldots, T_n$ subject to the quadratic and braid relations from Lemma 3.7. In particular, $\mathcal{H}$ is isomorphic to the Iwahori-Hecke algebra of* $\mathrm{SO}_{2n+1}(\mathbb{Q}_2)$.

Proof. Let A be the Iwahori-Hecke algebra of $\mathrm{SO}_{2n+1}(\mathbb{Q}_2)$ with generators $t_0, \ldots, t_n$. Then $(t_i - 2)(t_i + 1) = 0$ for all i and the braid relations satisfied by the t_i are given by the following Coxeter diagram.



For each $w \in W^{\mathrm{aff}}$, define an element $t_w = t_{i_1} \ldots t_{i_r}$ where $w = s_{i_1} \ldots s_{i_r}$ is a minimal expression for w. By [5], $\{t_w : w \in W^{\mathrm{aff}}\}$ forms a basis for A.

Let τ be the involution that exchanges the t_0 and t_1 vertices of the Coxeter diagram. The automorphism τ can replace t_0 as a generator of A (since $t_0 = \tau\, t_1 \tau$) which satisfies

$$\tau^2 = 1 \qquad \text{and} \qquad \tau\, t_1 \tau\, t_1 = t_0 t_1 = t_1 t_0 = t_1 \tau\, t_1 \tau.$$

Define a map on the generators from A to $\mathcal{H}$ by

$$\tau \mapsto T_n, \qquad \text{and} \qquad t_i \mapsto T_{n-i} \quad \text{for } i = 1, \ldots, n.$$

This map extends to a homomorphism which sends each basis element t_w of A to a basis element T_w of $\mathcal{H}$, so it must be an isomorphism. $\square$

References

[1] R. Carter, *Simple Groups of Lie Type*, Wiley, 1989.

[2] W.T. Gan, G. Savin, *Irreducible Representations of Metaplectic Groups II: Hecke Algebra Correspondences*, to appear in Representation Theory.

[3] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1997.

[4] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, 1972.

[5] N. Iwahori, H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic reductive groups*, Pub. Math. IHES, 25 (1965), pp. 5-48.

[6] H.Y. Loke, G. Savin, *Modular forms on nonlinear covers of algebraic groups*, Transactions of the American Mathematical Society, 362 (2010), pp.4901-4920.

[7] H.Y. Loke, G. Savin, *Representations of the two-fold central extension of* $\mathrm{SL}_2(\mathbb{Q}_2)$, Pacific Journal of Mathematics, 247:2 (2010), pp. 435-455.

[8] J.-P. Serre, *Linear Representations of Finite Groups*, Springer, 1977.

[9] R. Steinberg, *Lectures on Chevalley Groups*, Yale University, 1968.

[10] L. Washington, *On the self-duality of* $\mathbb{Q}_p$, The American Mathematical Monthly, 81:4 (1974), pp. 369-371.

[11] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. 111 (1964), pp. 143-211.

Aaron Wood, Department of Mathematics, University of Utah, Salt Lake City, UT 84112
Email: wood@math.utah.edu